Solve the differential equation $\dfrac{d^2y}{dx^2} = \sec^2 y \tan y$, with the initial conditions $y(-1)=0$ and $y'(-1)=1$. [106 台大機械 2]

[解]令 $p = y' \Rightarrow y'' = \dfrac{dy'}{dx} = \dfrac{dy'}{dy}\dfrac{dy}{dx} = \dfrac{dp}{dy}p = p\dfrac{dp}{dy}$，原式為

$$p\frac{dp}{dy} = \sec^2 y\tan y \Rightarrow pdp = \sec^2 y\tan ydy \Rightarrow \int pdp = \int \sec^2 y\tan ydy + k_1$$

$$\frac{1}{2}p^2 = \int \tan yd\tan y + k_1 \Rightarrow \frac{1}{2}p^2 = \frac{1}{2}\tan^2 y + k_1 \Rightarrow p^2 = \tan^2 y + C_1 \cdots\cdots\cdots(\text{i})$$

$y(-1)=0$ 及 $y'(-1)=1$ 代入(i) $\Rightarrow 1^2 = \tan^2 0 + C_1 \Rightarrow C_1 = 1$

$$(\text{i}) \Rightarrow p^2 = \tan^2 y + 1 \Rightarrow p = \sqrt{\tan^2 y + 1} = \sqrt{\sec^2 y} = \sec y \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\cos ydy = dx \Rightarrow \int \cos ydy = \int dx + C_2 \Rightarrow \sin y = x + C_2 \cdots\cdots\cdots(\text{ii})$$

$y(-1)=0$ 代入(ii) $\Rightarrow \sin 0 = -1 + C_2 \Rightarrow C_2 = 1$

$(\text{ii}) \Rightarrow \sin y = x + 1$