Show that the Fourier series of $f(x)=x$，$-\pi<x<\pi$ leads to $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$. [97 中央機械能源光機電生醫 8]

[解] $f(x)$為奇函數 $\Rightarrow$ 令 $f(x)=\sum_{n=1}^{\infty} b_n \sin nx$

$$b_n=\frac{2}{\pi}\int_0^{\pi} x\sin nx dx=-\frac{2}{n\pi}(x\cos nx\Big|_0^{\pi}-\int_0^{\pi}\cos nx dx)=-\frac{2}{n\pi}(\pi\cos n\pi)=\frac{2(-1)^{n-1}}{n}$$

$f(x)=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx$，$x=\frac{\pi}{2}$代入得 $\frac{f(\frac{\pi}{2}^{-})+f(\frac{\pi}{2}^{+})}{2}=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{n\pi}{2}$

$$\frac{\pi}{2}=2(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots)\Rightarrow\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$