Solve the ordinary differential equation $4y''-4y'+y=e^{x/2}\sqrt{1-x^2}$. [99 清大動機 1b]

[解] $4\lambda^2-4\lambda+1=0\Rightarrow\lambda=1/2,\ 1/2\Rightarrow y_h(x)=(C_1+C_2x)e^{x/2}\Rightarrow$ 令 $y_p(x)=(u_1+u_2x)e^{x/2}$

$y_p'=[(u_1/2+u_2+u_2x/2)+(u_1'+u_2'x)]e^{x/2}\Rightarrow$ 令 $u_1'+u_2'x=0\Rightarrow u_1'+xu_2'=0\cdots\cdots\cdots$(i)

$y_p''=[(u_1/4+u_2+u_2x/4)+(u_1'/2+u_2'+u_2'x/2)]e^{x/2}$

代入原式：$4[(u_1/4+u_2+u_2x/4)+(u_1'/2+u_2'+u_2'x/2)]e^{x/2}-4(u_1/2+u_2+u_2x/2)e^{x/2}$

$+(u_1+u_2x)e^{x/2}=e^{x/2}\sqrt{1-x^2}$

$2u_1'+(2x+4)u_2'=\sqrt{1-x^2}\cdots\cdots\cdots$(ii)

由(i)與(ii)得

$$\Delta=\begin{vmatrix}1 & x\\ 2 & 2x+4\end{vmatrix}=4,\ \Delta_1=\begin{vmatrix}0 & x\\ \sqrt{1-x^2} & 2x+4\end{vmatrix}=-x\sqrt{1-x^2},\ \Delta_2=\begin{vmatrix}1 & 0\\ 2 & \sqrt{1-x^2}\end{vmatrix}=\sqrt{1-x^2}$$

$$u_1'=\frac{\Delta_1}{\Delta}=\frac{-x\sqrt{1-x^2}}{4}\Rightarrow u_1=\int\frac{-x\sqrt{1-x^2}}{4}dx=\frac{1}{8}\int\sqrt{1-x^2}\,d(1-x^2)=\frac{1}{12}(1-x^2)^{3/2}$$

$$u_2'=\frac{\Delta_2}{\Delta}=\frac{\sqrt{1-x^2}}{4}\Rightarrow u_2=\int\frac{\sqrt{1-x^2}}{4}dx=\frac{1}{4}\int\cos\theta d\sin\theta=\frac{1}{4}\int\cos^2\theta d\theta=\frac{1}{8}\int(1+\cos2\theta)d\theta$$

$$=\frac{1}{8}\theta+\frac{\sin2\theta}{16}=\frac{1}{8}\theta+\frac{\sin\theta\cos\theta}{8}=\frac{1}{8}(\sin^{-1}x+x\sqrt{1-x^2})$$

$$y=y_h+y_p=[(C_1+C_2x)+\frac{1}{12}(1-x^2)^{3/2}+\frac{x}{8}(\sin^{-1}x+x\sqrt{1-x^2})]e^{x/2}$$