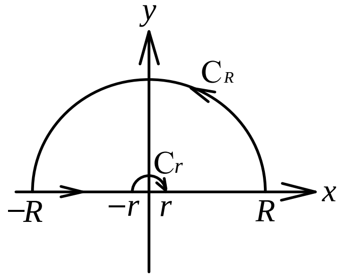
Evaluate $\int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+k^2)} dx$, $(m \geq 0,\ k > 0)$. [106 中興土木乙 7]

[解]令 $f(z) = \dfrac{e^{imz}}{z(z^2+k^2)} = \dfrac{e^{imz}}{z^3+k^2 z}$

$f(z)$有單極點在 $z = ki, 0$，其中0恰巧在實數軸上，須使用避點圍線



$$R_0 = \left.\frac{e^{imz}}{3z^2+k^2}\right|_{z=0} = \frac{1}{k^2} \qquad R_{ki} = \left.\frac{e^{imz}}{3z^2+k^2}\right|_{z=ki} = \frac{e^{-mk}}{-3k^2+k^2} = -\frac{e^{-mk}}{2k^2}$$

$$\int_{-R}^{-r} \frac{e^{imx}}{x(x^2+k^2)} dx + \int_{C_r} \frac{e^{imz}}{z(z^2+k^2)} dz + \int_{r}^{R} \frac{e^{imx}}{x(x^2+k^2)} dx$$

$$+ \int_{C_R} \frac{e^{imz}}{z(z^2+k^2)} dz = 2\pi i \cdot R_{ki}$$

$$\lim_{\substack{r\to 0\\ R\to\infty}} \int_{-R}^{-r} \frac{e^{imx}}{x(x^2+k^2)} dx - \pi i \cdot R_0 + \lim_{\substack{r\to 0\\ R\to\infty}} \int_{r}^{R} \frac{e^{imx}}{x(x^2+k^2)} dx + 0 = 2\pi i \cdot \left(-\frac{e^{-mk}}{2k^2}\right)$$

$$\int_{-\infty}^{0} \frac{e^{imx}}{x(x^2+k^2)} dx - \pi i \cdot \frac{1}{k^2} + \int_{0}^{\infty} \frac{e^{imx}}{x(x^2+k^2)} dx + 0 = -\frac{\pi i e^{-mk}}{k^2}$$

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x(x^2+k^2)} dx = \frac{\pi i}{k^2} - \frac{\pi i e^{-mk}}{k^2}$$

$$\int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{x(x^2+k^2)} dx = \frac{i\pi}{k^2}(1-e^{-mk})$$

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x(x^2+k^2)} dx = 0, \quad \int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+k^2)} dx = \frac{\pi}{k^2}(1-e^{-mk})$$