已知線積分$\int 2xydx + x^2dy$，積分路徑：$y = 3x - 2,\ 1 \le x \le 2$。(1)證明此積分是否與路徑有關？(2)求積分值。[104 高應大機械甲丙 7]

[解]設$\mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j} \Rightarrow F_1 = 2xy,\ F_2 = x^2,\ F_3 = 0$，則該線積分為$\int \mathbf{F} \cdot d\mathbf{r}$

(1)$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy & x^2 & 0 \end{vmatrix} = 2x\mathbf{k} - 2x\mathbf{k} = 0 \Rightarrow$ 此線積分與路徑無關

(2)存在一純量函數$\phi$，使得$\nabla\phi = \mathbf{F} \Rightarrow \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$

$$\frac{\partial\phi}{\partial x} = F_1 \Rightarrow \phi = \int_x F_1 dx = \int_x 2xydx = x^2y + f(y,z)$$

$$\frac{\partial\phi}{\partial y} = F_2 \Rightarrow x^2 + \frac{\partial f}{\partial y} = x^2 \Rightarrow \frac{\partial f}{\partial y} = 0 \Rightarrow f = g(z)$$

$$\frac{\partial\phi}{\partial z} = F_3 \Rightarrow \frac{dg}{dz} = 0 \Rightarrow g(z) = C$$

得$\phi = x^2y + C$，線積分從A(1, 1)到B(2, 4)

線積分$\int \mathbf{F} \cdot d\mathbf{r} = \int_{\mathrm{A}}^{\mathrm{B}} d\phi = \phi(\mathrm{A}) - \phi(\mathrm{B}) = 1^2 \cdot 1 - 2^2 \cdot 4 = 1 - 16 = -15$