Evaluate the surface integral $\oiint_S (x-z)dydz+(2y-z)dzdx-(2x-y)dxdy$ on the surface of the sphere S: $x^2+y^2+z^2=9$. [102 彰師大電機 4]

[解](1)Let $f=x^2+y^2+z^2$，S的單位法向量為

$$\mathbf{n}=\frac{\nabla f}{|\nabla f|}=\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}=\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}=\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{3}$$

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=\iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}$$

$$=\iint_S [(x-z)\mathbf{i}+(2y-z)\mathbf{j}-(2x-y)\mathbf{k}]\cdot\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{3}\frac{dxdy}{|\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{3}\cdot\mathbf{k}|}$$

$$=\iint_S [x(x-z)+y(2y-z)-z(2x-y)]\frac{dxdy}{|z|}=\iint_S (x^2+2y^2-3xz)\frac{dxdy}{|z|}$$

其中$3xz$為x的奇函數$\Rightarrow \iint_S -3xz\frac{dxdy}{|z|}=0$

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=\iint_S \frac{x^2+2y^2}{|z|}dxdy=8\int_0^3\int_0^{\sqrt{9-x^2}}\frac{x^2+2y^2}{\sqrt{9-(x^2+y^2)}}dxdy$$

令 $x=r\cos\theta,\ y=r\sin\theta$, 則

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=8\int_0^{\pi/2}\int_0^3\frac{r^2(1+\sin^2\theta)}{\sqrt{9-r^2}}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|drd\theta=8\int_0^{\pi/2}\int_0^3\frac{r^2(1+\sin^2\theta)}{\sqrt{9-r^2}}rdrd\theta$$

令 $r=3\sin\phi\Rightarrow dr=3\cos\phi d\phi,\ \sqrt{9-r^2}=3\cos\phi$

$$\iint_S \mathbf{F}\cdot d\mathbf{s}=8\int_0^{\pi/2}\int_0^{\pi/2}\frac{(3\sin\phi)^3}{3\cos\phi}(3\cos\phi d\phi)(1+\sin^2\theta)d\theta=8\int_0^{\pi/2}\int_0^{\pi/2}27\sin^3\phi d\phi(1+\sin^2\theta)d\theta$$

$$=8\int_0^{\pi/2}\int_0^{\pi/2}27\sin^2\phi(\sin\phi d\phi)(1+\sin^2\theta)d\theta$$

$$=8\int_0^{\pi/2}\int_0^{\pi/2}-27(1-\cos^2\phi)d(\cos\phi)(1+\sin^2\theta)d\theta$$

$$=8\int_0^{\pi/2}-27(\cos\phi-\frac{\cos^3\phi}{3})\Big|_0^{\pi/2}(1+\sin^2\theta)d\theta=8\cdot 18\int_0^{\pi/2}(1+\sin^2\theta)d\theta$$

$$=144\int_0^{\pi/2}(1+\frac{1-\cos 2\theta}{2})d\theta=144\int_0^{\pi/2}\frac{3-\cos 2\theta}{2}d\theta=144(\frac{3}{2}\theta-\frac{\sin 2\theta}{4})\Big|_0^{\pi/2}=108\pi$$

(2)$\mathbf{F}=(x-z)\mathbf{i}+(2y-z)\mathbf{j}-(2x-y)\mathbf{k}\Rightarrow\nabla\cdot\mathbf{F}=3$，由散度定理知

$$\oiint_S \mathbf{F}\cdot d\mathbf{s}=\iiint_V \nabla\cdot\mathbf{F}dv=\iiint_V 3dv=3\cdot(\frac{4}{3}\pi\cdot 3^3)=108\pi$$