Given a system of equations as below $\begin{cases} \dot{x}_1 = x_2 & \text{with } x_1(0)=0 \\ \dot{x}_2 = x_3 & x_2(0)=0 \\ \dot{x}_3 = -x_1 - 3x_2 - 3x_3 & x_3(0)=1 \end{cases}$. Solve the above system of equations for $x_1(t)$, $x_2(t)$, and $x_3(t)$. [90 雲科大機械 2]

[解]原式為 $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \dfrac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \cdots\cdots\cdots \text{(i)}$，其中 $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & -3 & -3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = -1, -1, -1$$

$$\lambda = -1, (\mathbf{A} - \lambda\mathbf{I})\mathbf{x}_1 = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x}_2^* = \mathbf{x}_1 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \mathbf{x}_2^* = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x}_3^* = \mathbf{x}_2^* \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \mathbf{x}_3^* = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \mathbf{J} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \mathbf{S}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -2 & -1 \end{bmatrix}$$

令 $\mathbf{x} = \mathbf{S}\mathbf{y}, \text{(i)} \Rightarrow \mathbf{S}\dfrac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{S}\mathbf{y} \Rightarrow \dfrac{d\mathbf{y}}{dt} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{y} \Rightarrow \dfrac{d\mathbf{y}}{dt} = \mathbf{J}\mathbf{y}$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\dot{y}_3 = -y_3 \Rightarrow y_3 = c_3 e^{-t}$$

$$\dot{y}_2 = -y_2 + y_3 = -y_2 + c_3 e^{-t} \Rightarrow y_2 = c_2 e^{-t} + c_3 t e^{-t}$$

$$\dot{y}_1 = -y_1 + y_2 = -y_1 + c_2 e^{-t} + c_3 t e^{-t} \Rightarrow y_1 = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{2}c_3 t^2 e^{-t}$$

$$\mathbf{x} = \mathbf{S}\mathbf{y} \Rightarrow \mathbf{x} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} (c_1 + c_2 t + \frac{1}{2}c_3 t^2)e^{-t} \\ (c_2 + c_3 t)e^{-t} \\ c_3 e^{-t} \end{bmatrix} = \begin{bmatrix} [-c_1 + c_2(-1-t) + c_3(-1-t-\frac{1}{2}t^2)]e^{-t} \\ (c_1 + c_2 t + \frac{1}{2}c_3 t^2)e^{-t} \\ [-c_1 + c_2(1-t) + c_3(t - \frac{1}{2}t^2)]e^{-t} \end{bmatrix}$$