Solve the following partial differential equation

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2} \quad \text{for } 0<x<L,\ t>0$$

$$u(0,t)=u(L,t)=0 \ \text{for} t\ge 0$$

$$u(x,0)=x(L-x) \ \text{for } 0\le x\le L$$ [102彰師大機電電子資工5]

[解]設$u(x,t)=X(x)T(t)$，代入統御方程式，得

$$XT'=kX''T \Rightarrow \frac{T'}{kT}=\frac{X''}{X}=-\lambda^2 \Rightarrow \begin{cases} X''+\lambda^2 X=0 \\ T'+k\lambda^2 T=0 \end{cases}$$

邊界條件$\begin{cases} u(0,t)=0 \Rightarrow X(0)T(t)=0 \Rightarrow X(0)=0 \\ u(L,t)=0 \Rightarrow X(L)T(t)=0 \Rightarrow X(L)=0 \end{cases}$

先解$X''+\lambda^2 X=0,\ X(0)=0,\ X(L)=0 \Rightarrow$ 得$X=C_1\cos\lambda x+C_2\sin\lambda x$

$$\begin{cases} X(0)=0 \Rightarrow C_1=0 \\ X(L)=0 \Rightarrow C_2\sin\lambda L=0 \end{cases} \Rightarrow \sin\lambda L=0 \Rightarrow \lambda_n L=n\pi \Rightarrow \lambda_n=\frac{n\pi}{L} \text{ 且 } X_n=\sin\lambda_n x$$

由$T'+k\lambda_n^2 T=0 \Rightarrow T_n=e^{-k\lambda_n^2 t} \Rightarrow u=\sum_{n=1}^{\infty} b_n e^{-k\lambda_n^2 t}\sin\lambda_n x$

初始條件$u(x,0)=x(L-x) \Rightarrow \sum_{n=1}^{\infty} b_n \sin\lambda_n x = x(L-x)$

$$b_n=\frac{2}{L}\int_0^L x(L-x)\sin\lambda_n x\,dx=\frac{2}{L}\int_0^L (Lx-x^2)\sin\frac{n\pi x}{L}dx$$

$$=\frac{2}{L}\cdot\frac{-L}{n\pi}\left[(Lx-x^2)\cos\frac{n\pi x}{L}\Big|_0^L-\int_0^L (L-2x)\cos\frac{n\pi x}{L}dx\right]$$

$$=\frac{2}{n\pi}\cdot\frac{L}{n\pi}\left[(L-2x)\sin\frac{n\pi x}{L}\Big|_0^L-\int_0^L (-2)\sin\frac{n\pi x}{L}dx\right]$$

$$=-\frac{4L^2}{n^3\pi^3}\cos\frac{n\pi x}{L}\Big|_0^L=-\frac{4L^2}{n^3\pi^3}[(-1)^n-1]=\frac{8L^2}{(2n-1)^3\pi^3}$$

$$u=\sum_{n=1}^{\infty}\frac{8L^2}{(2n-1)^3\pi^3}e^{-\frac{k(2n-1)^2\pi^2 t}{L^2}}\sin\frac{(2n-1)\pi x}{L}$$