Suppose that $\mathbf{A}=\begin{bmatrix}1&1&0\\1&1&0\\0&0&3\end{bmatrix}$ (a)show that $e^{\mathbf{A}t}=\mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}$, where $\mathbf{A}=\mathbf{PDP}^{-1}$ with $\mathbf{D}$ being a diagonal matrix of diagonal terms being the eigenvalues of $\mathbf{A}$. (b)Use the result from (a) to find the general solution of $\mathbf{X}'=\begin{bmatrix}1&1&0\\1&1&0\\0&0&3\end{bmatrix}\mathbf{X}+\begin{bmatrix}e^{t}\\e^{2t}\\te^{3t}\end{bmatrix}$. [103 中興電機光電 8]

[解](a) $|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}1-\lambda&1&0\\1&1-\lambda&0\\0&0&3-\lambda\end{vmatrix}=0\Rightarrow-(\lambda-1)^2(\lambda-3)+(\lambda-3)=0$

$$(\lambda-3)[(\lambda-1)^2-1]=0\Rightarrow\lambda(\lambda-2)(\lambda-3)=0\Rightarrow\lambda=0,2,3$$

$$\lambda=0,\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\Rightarrow\begin{bmatrix}1&1&0\\1&1&0\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0\Rightarrow\mathbf{x}_1=\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$\lambda=2,\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\Rightarrow\begin{bmatrix}-1&1&0\\1&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0\Rightarrow\mathbf{x}_2=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\lambda=3,\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\Rightarrow\begin{bmatrix}-2&1&0\\1&-2&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0\Rightarrow\mathbf{x}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\mathbf{D}=\begin{bmatrix}0&0&0\\0&2&0\\0&0&3\end{bmatrix},\ \mathbf{P}=\begin{bmatrix}1&1&0\\-1&1&0\\0&0&1\end{bmatrix}\Rightarrow\mathbf{P}^{-1}=\frac{1}{2}\begin{bmatrix}1&-1&0\\1&1&0\\0&0&2\end{bmatrix}$$

$$e^{\mathbf{A}t}=\mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}=\begin{bmatrix}1&1&0\\-1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&e^{2t}&0\\0&0&e^{3t}\end{bmatrix}\frac{1}{2}\begin{bmatrix}1&-1&0\\1&1&0\\0&0&2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&e^{2t}&0\\-1&e^{2t}&0\\0&0&e^{3t}\end{bmatrix}\begin{bmatrix}1&-1&0\\1&1&0\\0&0&2\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}1+e^{2t}&-1+e^{2t}&0\\-1+e^{2t}&1+e^{2t}&0\\0&0&2e^{3t}\end{bmatrix}$$

(b)令 $\mathbf{X}=\mathbf{P}\mathbf{Y}\Rightarrow\mathbf{X}'=\mathbf{P}\mathbf{Y}'\Rightarrow\mathbf{Y}'=\mathbf{P}^{-1}\mathbf{X}'\Rightarrow\mathbf{Y}'=\mathbf{P}^{-1}(\mathbf{A}\mathbf{X}+\mathbf{h})$, 其中 $\mathbf{h}=\begin{bmatrix}e^t\\e^{2t}\\te^{3t}\end{bmatrix}$

$$\mathbf{Y}'=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{Y}+\mathbf{P}^{-1}\mathbf{h}=\mathbf{D}\mathbf{Y}+\mathbf{P}^{-1}\mathbf{h}\Rightarrow\mathbf{Y}=e^{\mathbf{D}t}[\int e^{-\mathbf{D}t}\mathbf{P}^{-1}\mathbf{h}dt+\mathbf{c}]\Rightarrow\mathbf{X}=\mathbf{P}e^{\mathbf{D}t}[\int e^{-\mathbf{D}t}\mathbf{P}^{-1}\mathbf{h}dt+\mathbf{c}]$$

$$\mathbf{X}=\begin{bmatrix}1&1&0\\-1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&e^{2t}&0\\0&0&e^{3t}\end{bmatrix}[\int\begin{bmatrix}1&0&0\\0&e^{-2t}&0\\0&0&e^{-3t}\end{bmatrix}\frac{1}{2}\begin{bmatrix}1&-1&0\\1&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}e^t\\e^{2t}\\te^{3t}\end{bmatrix}dt+\mathbf{c}]$$

$$=\frac{1}{2}\begin{bmatrix}1&e^{2t}&0\\-1&e^{2t}&0\\0&0&e^{3t}\end{bmatrix}[\int\begin{bmatrix}1&-1&0\\e^{-2t}&e^{-2t}&0\\0&0&2e^{-3t}\end{bmatrix}\begin{bmatrix}e^t\\e^{2t}\\te^{3t}\end{bmatrix}dt+\mathbf{c}]$$

$$=\frac{1}{2}\begin{bmatrix}1&e^{2t}&0\\-1&e^{2t}&0\\0&0&e^{3t}\end{bmatrix}[\int\begin{bmatrix}e^t-e^{2t}\\e^{-t}+1\\2\end{bmatrix}dt+\mathbf{c}]=\frac{1}{2}\begin{bmatrix}1&e^{2t}&0\\-1&e^{2t}&0\\0&0&e^{3t}\end{bmatrix}[\begin{bmatrix}e^t-\frac{1}{2}e^{2t}\\-e^{-t}+t\\2t\end{bmatrix}+\mathbf{c}]$$

$$=\frac{1}{2}\begin{bmatrix}1&e^{2t}&0\\-1&e^{2t}&0\\0&0&e^{3t}\end{bmatrix}\begin{bmatrix}C_1+e^t-\frac{1}{2}e^{2t}\\C_2-e^{-t}+t\\C_3+2t\end{bmatrix}=\frac{1}{2}\begin{bmatrix}C_1+C_2e^{2t}-\frac{1}{2}e^{2t}+te^{2t}\\-C_1+C_2e^{2t}-2e^t+\frac{1}{2}e^{2t}+te^{2t}\\C_3e^{3t}+2te^{3t}\end{bmatrix}$$