Given a vector field $\mathbf{F}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, and a surface defined by S: $x^2+y^2+z^2=1$, $z\geq 0$. (1)Calculate $\iint_S \mathbf{F}\cdot\mathbf{n}\,ds$, where $\mathbf{n}$ is an outward normal unit vector. (2)If $\mathbf{F}=-\nabla U$, find $U(x, y, z)$. [104 高應大土木 3(3)(4)]

[解]$\mathbf{F}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \Rightarrow F_1=x, F_2=y, F_3=z$

(1)設 $f=x^2+y^2+z^2$，S的單位法向量為

$$\mathbf{n}=\frac{\nabla f}{|\nabla f|}=\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}=\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}}=\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\cdot 1}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}ds=\iint_S \mathbf{F}\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}=\iint_S (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\frac{dxdy}{|(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{k}|}$$

$$=\iint_S (x^2+y^2+z^2)\frac{dxdy}{z}=4\int_0^1\int_0^{\sqrt{1-x^2}}\frac{1}{\sqrt{1-(x^2+y^2)}}dydx$$

令 $x=r\cos\theta, y=r\sin\theta$，則

$$\iint_S \mathbf{F}\cdot\mathbf{n}ds=4\int_0^{\pi/2}\int_0^1\frac{1}{\sqrt{1-r^2}}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|drd\theta=4\int_0^{\pi/2}\int_0^1\frac{1}{\sqrt{1-r^2}}r\,drd\theta$$

$$=4\int_0^{\pi/2}-\frac{1}{2}\cdot 2(1-r^2)^{1/2}\Big|_0^1 d\theta=4\int_0^{\pi/2}d\theta=4\theta\Big|_0^{\frac{\pi}{2}}=4\cdot\frac{\pi}{2}=2\pi$$

(2)$\mathbf{F}=-\nabla U \Rightarrow F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}=-(\frac{\partial U}{\partial x}\mathbf{i}+\frac{\partial U}{\partial y}\mathbf{j}+\frac{\partial U}{\partial z}\mathbf{k})$

$$F_1=-\frac{\partial U}{\partial x}\Rightarrow U=-\int_x F_1dx=-\int_x xdx=-\frac{x^2}{2}+f(y,z)$$

$$F_2=-\frac{\partial U}{\partial y}\Rightarrow y=-\frac{\partial f}{\partial y}\Rightarrow f=-\int_y ydy+g(z)=-\frac{y^2}{2}+g(z)$$

$$F_3=-\frac{\partial U}{\partial z}\Rightarrow z=-\frac{dg}{dz}\Rightarrow g(z)=-\int zdz+C=-\frac{z^2}{2}+C$$

得$U=-\frac{x^2}{2}-\frac{y^2}{2}-\frac{z^2}{2}+C$