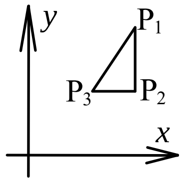
Given a vector function $\mathbf{F}=xy\mathbf{i}+(3x-y^2)\mathbf{j}$, and the coordinates of three points $P_1(5, 6)$, $P_2(5, 3)$, $P_3(3, 3)$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along x- and y- axes respectively. (1)Evaluate the integral $\int \mathbf{F}\cdot d\mathbf{r}$ from P_1 straight to P_3. (2)Evaluate the integral $\int \mathbf{F}\cdot d\mathbf{r}$ from P_1 to P_3 along the piecewise straight path $P_1P_2P_3$, i.e., integrate from P_1 along straight line segment to P_2 and then along another straight line from P_2 to P_3. (3)Is this $\mathbf{F}$ a conservative field? And why? [99 清大動機甲丙丁 4]



[解]$\mathbf{F}=xy\mathbf{i}+(3x-y^2)\mathbf{j}\Rightarrow F_1=xy,\quad F_2=3x-y^2, F_3=0$

(1)連接P_1P_3的直線為 $y-6=\dfrac{6-3}{5-3}(x-5)\Rightarrow 3x-2y=3$

設 $x=2t+1, y=3t\Rightarrow d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}=(2\mathbf{i}+3\mathbf{j})dt$

$\mathbf{F}=xy\mathbf{i}+(3x-y^2)\mathbf{j}=(2t+1)\cdot 3t\mathbf{i}+[3(2t+1)-(3t)^2]\mathbf{j}=(6t^2+3t)\mathbf{i}+(-9t^2+6t+3)\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C[(6t^2+3t)\mathbf{i}+(-9t^2+6t+3)\mathbf{j}]\cdot(2\mathbf{i}+3\mathbf{j})dt=\int_2^1[2(6t^2+3t)+3(-9t^2+6t+3)]dt$$

$$=\int_2^1(-15t^2+24t+9)dt=(-5t^3+12t^2+9t)\Big|_2^1=-10$$

(2)C_1: 連接P_1P_2的線段 $x=5,\ dx=0,\ d\mathbf{r}=dy\mathbf{j}$

$\mathbf{F}=xy\mathbf{i}+(3x-y^2)\mathbf{j}=5y\mathbf{i}+(15-y^2)\mathbf{j}$

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r}=\int_{C_1}[5y\mathbf{i}+(15-y^2)\mathbf{j}]\cdot dy\mathbf{j}=\int_6^3(15-y^2)dy=(15y-\frac{y^3}{3})\Big|_6^3=18$$

C_2: 連接P_2P_3的線段 $y=3,\ dy=0,\ d\mathbf{r}=dx\mathbf{i}$

$\mathbf{F}=xy\mathbf{i}+(3x-y^2)\mathbf{j}=3x\mathbf{i}+(3x-9)\mathbf{j}$

$$\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\int_{C_2}[3x\mathbf{i}+(3x-9)\mathbf{j}]\cdot dx\mathbf{i}=\int_5^3 3xdx=\frac{3x^2}{2})\Big|_5^3=-24$$

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_{C_1}\mathbf{F}\cdot d\mathbf{r}+\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=18+(-24)=-6$$

(3)$\mathbf{F}$不是保守場，因為線積分值與路徑有關。