(a)Show that $x = \exp(z)$ then $x\dfrac{dy}{dx} = \dfrac{dy}{dz}$ and $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2}$.

(b)Solve the Euler equation $x^2y'' + 2xy' - 12y = \sqrt{x}$, $x > 0$ by using the method of transformation you have verified in part (a). [85 清大動機 1]

[解](a) $\dfrac{d}{dx} = \dfrac{dz}{dx}\dfrac{d}{dz} = \dfrac{1}{e^z}\dfrac{d}{dz} \Rightarrow x\dfrac{dy}{dx} = e^z \cdot \dfrac{1}{e^z}\dfrac{dy}{dz} = \dfrac{dy}{dz}$

$$x^2\frac{d^2}{dx^2} = x^2\frac{d}{dx}\left(\frac{d}{dx}\right) = e^{2z}\frac{1}{e^z}\frac{d}{dz}\left(\frac{1}{e^z}\frac{d}{dz}\right) = e^z\left(\frac{1}{e^z}\frac{d^2}{dz^2} - \frac{1}{e^z}\frac{d}{dz}\right) = \frac{d^2}{dz^2} - \frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

(b)原式 $\Rightarrow (\dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}) + 2\dfrac{dy}{dz} - 12y = e^{\frac{1}{2}z} \Rightarrow \dfrac{d^2y}{dz^2} + \dfrac{dy}{dz} - 12y = e^{\frac{1}{2}z}$ ………(i)

特徵方程式為 $\lambda^2 + \lambda - 12 = 0 \Rightarrow (\lambda + 4)(\lambda - 3) = 0 \Rightarrow \lambda = -4, 3 \Rightarrow y_h = C_1e^{-4z} + C_2e^{3z}$

令 $y_p = Ae^{\frac{1}{2}z} \Rightarrow \dfrac{dy_p}{dz} = \dfrac{1}{2}Ae^{\frac{1}{2}z}, \dfrac{d^2y_p}{dz^2} = \dfrac{1}{4}Ae^{\frac{1}{2}z}$

(i) $\Rightarrow \dfrac{1}{4}Ae^{\frac{1}{2}z} + \dfrac{1}{2}Ae^{\frac{1}{2}z} - 12Ae^{\frac{1}{2}z} = e^{\frac{1}{2}z} \Rightarrow A = -\dfrac{4}{45}$

解為 $y = y_h + y_p = C_1e^{-4z} + C_2e^{3z} - \dfrac{4}{45}e^{\frac{1}{2}z} = \dfrac{C_1}{x^4} + C_2x^3 - \dfrac{4}{45}\sqrt{x}$