Please solve the partial differential equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\sin x,\text{ with }\begin{cases}u(x,0)=400+\sin x,\ 0<x<\pi\\ t>0,\ u(0,t)=400,\ u(\pi,t)=200\end{cases}\quad [105成大環工VI]$$

[解]當使用分離變數法在齊性邊界條件$u(0,t)=0, u(\pi,t)=0$，得特徵函數$\sin nx$

將方程式對x取有限正弦轉換

$$\frac{2}{\pi}\int_0^\pi (u_t-u_{xx})\sin nxdx=\frac{2}{\pi}\int_0^\pi \sin x\sin nxdx$$

$$\frac{2}{\pi}\int_0^\pi u_t\sin nxdx-\frac{2}{\pi}\int_0^\pi u_{xx}\sin nxdx=1$$

$$\frac{d}{dt}\frac{2}{\pi}\int_0^\pi u\sin nxdx-\frac{2}{\pi}\left[u_x\sin nx\Big|_0^\pi-n\int_0^\pi u_x\cos nxdx\right]=1$$

$$\frac{dU_n}{dt}-\frac{2}{\pi}\left[0-n\left(u\cos nx\Big|_0^\pi+n\int_0^\pi u\sin nxdx\right)\right]=1$$

$$\frac{dU_n}{dt}-\frac{2}{\pi}\left[-n(-200-400)-n^2\int_0^\pi u\sin nxdx\right]=1$$

$$\frac{dU_n}{dt}+n^2U_n=\frac{1200n}{\pi}+1\cdots\cdots\cdots\text{(i)}$$

其中$u(x,t)=\sum_{n=1}^{\infty}U_n(t)\sin nx\Rightarrow U_n(t)=\frac{2}{\pi}\int_0^\pi u(x,t)\sin nxdx$

$$\int_0^\pi \sin x\sin nxdx=\begin{cases}0,\ n\neq 1\\ \frac{\pi}{2},\ n=1\end{cases}$$

(i)的解為$U_n=e^{-n^2t}[\int e^{n^2t}(\frac{1200n}{\pi}+1)dt+U_n(0)]\cdots\cdots\cdots$(ii)

其中$U_n(0)=\frac{2}{\pi}\int_0^\pi (400+\sin x)\sin nxdx=-\frac{800\cos nx}{n\pi}\Big|_0^\pi+1=\frac{1600}{n\pi}+1$

$$u(x,t)=\sum_{n=1}^{\infty}[\frac{1}{n^2}(\frac{1200n}{\pi}+1)+(\frac{1600}{n\pi}+1)e^{-n^2t}]\sin nx$$