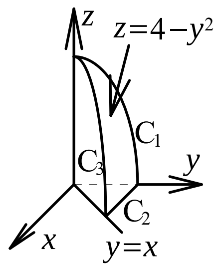
Verify Stoke's theorem, by calculating both sides of the equation, for the case $\mathbf{v}=xz\mathbf{j}$, and S is the surface $z=4-y^2$ cut off by the planes $x=0$, $z=0$ and $y=x$. [103 台科大機械 4]



[解](1)令 $f=y^2+z \Rightarrow \mathbf{n}=\dfrac{\nabla f}{|\nabla f|}=\dfrac{2y\mathbf{j}+\mathbf{k}}{\sqrt{(2y)^2+1}}=\dfrac{2y\mathbf{j}+\mathbf{k}}{\sqrt{4y^2+1}}$

$$\nabla\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 0 & xz & 0\end{vmatrix}=-x\mathbf{i}+z\mathbf{k}$$

$$(\nabla\times\mathbf{v})\cdot\mathbf{n}=\frac{z}{\sqrt{4y^2+1}},\ \mathbf{n}\cdot\mathbf{k}=\frac{1}{\sqrt{4y^2+1}}$$

$$\iint_S(\nabla\times\mathbf{v})\cdot d\mathbf{s}=\iint_S(\nabla\times\mathbf{v})\cdot\mathbf{n}ds=\int_0^2\int_0^y(\nabla\times\mathbf{v})\cdot\mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}$$

$$=\int_0^2\int_0^y zdxdy=\int_0^2\int_0^y(4-y^2)dxdy=\int_0^2(4x-xy^2)\Big|_0^y dy$$

$$=\int_0^2(4y-y^3)dy=(2y^2-\frac{y^4}{4})\Big|_0^2=4$$

(2)在 C_1 上 $x=0$、在 C_2 上 $z=0\Rightarrow\mathbf{v}=0$

在 C_3 上 $z=4-y^2,\ y=x\Rightarrow\mathbf{v}=xz\mathbf{j}=y(4-y^2)\mathbf{j}$

$$\oint_C\mathbf{v}\cdot d\mathbf{r}=\int_{C_1}\mathbf{v}\cdot d\mathbf{r}+\int_{C_2}\mathbf{v}\cdot d\mathbf{r}+\int_{C_3}\mathbf{v}\cdot d\mathbf{r}$$

$$=0+0+\int_0^2 y(4-y^2)\mathbf{j}\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$

$$=\int_0^2 y(4-y^2)dy=(2y^2-\frac{y^4}{4})\Big|_0^2=4$$