Transform the quadratic form $7x_1^2 + 5x_2^2 + 6x_3^2 - 4x_1x_3 + 4x_2x_3 = 210$ to principle axes (to a canonical form; i.e. the form of $\lambda_1x_1^2+\lambda_2x_2^2+\lambda_3x_3^2=210$). Express the new coordinate vector $\mathbf{y}=[y_1 \quad y_2 \quad y_3]^T$ in terms of the original coordinate vector $\mathbf{x}=[x_1 \quad x_2 \quad x_3]^T$. [104 中山光電 2]

[解]令$Q = 7x_1^2 + 5x_2^2 + 6x_3^2 - 4x_1x_3 + 4x_2x_3 = \mathbf{x}^T\mathbf{A}\mathbf{x}$

其中$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$，$\mathbf{A} = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 2 \\ -2 & 2 & 6 \end{bmatrix}$，即$Q = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 2 \\ -2 & 2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{vmatrix} 7-\lambda & 0 & -2 \\ 0 & 5-\lambda & 2 \\ -2 & 2 & 6-\lambda \end{vmatrix} = 0 \Rightarrow -(\lambda-7)(\lambda-5)(\lambda-6)+4(\lambda-5)+4(\lambda-7)=0$$

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0 \Rightarrow (\lambda-3)(\lambda-6)(\lambda-9) = 0 \Rightarrow \lambda = 3, 6, 9$$

$$\lambda = 3, \begin{bmatrix} 4 & 0 & -2 \\ 0 & 2 & 2 \\ -2 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \Rightarrow \mathbf{e}_1 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$\lambda = 6, \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{e}_2 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda = 9, \begin{bmatrix} -2 & 0 & -2 \\ 0 & -4 & 2 \\ -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \mathbf{x}_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \mathbf{e}_3 = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

令$\mathbf{x} = \mathbf{M}\mathbf{y}$，其中$\mathbf{M} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$，則

$$Q = \mathbf{y}^T\mathbf{M}^T\mathbf{A}\mathbf{M}\mathbf{y} = 3y_1^2 + 6y_2^2 + 9y_3^2$$

$$\mathbf{y} = \mathbf{M}^T\mathbf{x} = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}\mathbf{x}$$