A curve is defined as $\mathbf{r}(t)=[a\cos t, a\sin t, ct]$, Please find $\mathbf{u}(s)$, where s is the arc length and $\mathbf{u}(s)$ is the unit tangent vector; $\kappa(s)$, where $\kappa(s)$ is the curvature of the curve; $\mathbf{p}(s)$, where $\mathbf{p}(s)$ is the unit principle normal vector; $\mathbf{b}(s)$, where $\mathbf{b}(s)$ is the unit binormal vector; $\tau(s)$, where $\tau(s)$ is the torsion of the curve. [96 暨南土木 4(b)(c)(d)(e)(f)]

[解] $\mathbf{v}=\dot{\mathbf{r}}=-a\sin t\mathbf{i}+a\cos t\mathbf{j}+c\mathbf{k}=v\mathbf{u}\Rightarrow v=\sqrt{a^2+c^2},\ \dot{v}=0$

$$s=\int_0^t v\,dt=\int_0^t\sqrt{a^2+c^2}\,dt=t\sqrt{a^2+c^2}\Rightarrow t=\frac{s}{\sqrt{a^2+c^2}}$$

$$\mathbf{u}=\frac{\mathbf{v}}{v}=\frac{-a\sin t}{\sqrt{a^2+c^2}}\mathbf{i}+\frac{a\cos t}{\sqrt{a^2+c^2}}\mathbf{j}+\frac{c}{\sqrt{a^2+c^2}}\mathbf{k}$$

$$=\frac{1}{\sqrt{a^2+c^2}}\left(-a\sin\frac{s}{\sqrt{a^2+c^2}}\mathbf{i}+a\cos\frac{s}{\sqrt{a^2+c^2}}\mathbf{j}+c\mathbf{k}\right)$$

$$\mathbf{a}=\ddot{\mathbf{r}}=-a\cos t\mathbf{i}-a\sin t\mathbf{j}=\dot{v}\mathbf{u}+\kappa v^2\mathbf{p}=\kappa(a^2+c^2)\mathbf{p}\Rightarrow\dddot{\mathbf{r}}=a\sin t\mathbf{i}-a\cos t\mathbf{j}$$

$$\dot{\mathbf{r}}\times\ddot{\mathbf{r}}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a\sin t&a\cos t&c\\-a\cos t&-a\sin t&0\end{vmatrix}=\kappa v^3\mathbf{b}\Rightarrow ac\sin t\mathbf{i}-ac\cos t\mathbf{j}+a^2\mathbf{k}=\kappa v^3\mathbf{b}$$

$$\kappa v^3=\sqrt{(ac\sin t)^2+(-ac\cos t)^2+(a^2)^2}=\sqrt{a^2c^2+a^4}=a\sqrt{a^2+c^2}\Rightarrow\kappa=\frac{a}{a^2+c^2}$$

代入$\mathbf{a}$: $-a\cos t\mathbf{i}-a\sin t\mathbf{j}=\dfrac{a}{a^2+c^2}\cdot(a^2+c^2)\mathbf{p}$

$$\mathbf{p}=-\cos t\mathbf{i}-\sin t\mathbf{j}=-\cos\frac{s}{\sqrt{a^2+c^2}}\mathbf{i}-\sin\frac{s}{\sqrt{a^2+c^2}}\mathbf{j}$$

$$\tau=\frac{(\dot{\mathbf{r}}\ \ddot{\mathbf{r}}\ \dddot{\mathbf{r}})}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2}=\frac{a^2c}{a^2(a^2+c^2)}=\frac{c}{a^2+c^2}$$

$$\mathbf{b}=\mathbf{u}\times\mathbf{p}=\frac{1}{\sqrt{a^2+c^2}}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a\sin t&a\cos t&c\\-\cos t&-\sin t&0\end{vmatrix}=\frac{1}{\sqrt{a^2+c^2}}(c\sin t\mathbf{i}-c\cos t\mathbf{j}+a\mathbf{k})$$

$$=\frac{1}{\sqrt{a^2+c^2}}\left(c\sin\frac{s}{\sqrt{a^2+c^2}}\mathbf{i}-c\cos\frac{s}{\sqrt{a^2+c^2}}\mathbf{j}+a\mathbf{k}\right)$$