Expand $f(x)=\begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$, in a Fourier series. [102 中原生醫 4、104 中原土木 3]

[解] $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{1}{\pi}\int_0^{\pi}(\pi - x)dx=\frac{1}{\pi}\left(\pi x-\frac{x^2}{2}\Big|_0^{\pi}\right)=\frac{\pi}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx=\frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos nx dx=\frac{1}{n\pi}\left[(\pi - x)\sin nx\Big|_0^{\pi}+\int_0^{\pi}\sin nx dx\right]$$

$$=\frac{1}{n\pi}\left[0-\frac{\cos nx}{n}\Big|_0^{\pi}\right)]=-\frac{\cos n\pi - 1}{n^2\pi}=-\frac{(-1)^n-1}{n^2\pi}=\frac{1-(-1)^n}{n^2\pi}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx dx=-\frac{1}{n\pi}\left[(\pi - x)\cos nx\Big|_0^{\pi}+\int_0^{\pi}\cos nx dx\right]$$

$$=-\frac{1}{n\pi}(-\pi+0)=\frac{1}{n}$$

$$f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2\pi}\cos nx+\frac{1}{n}\sin nx\right],\quad -\pi < x < \pi$$