Solve the first order differential system $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. [89 暨南電機 2]

[解]令 $\mathbf{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix}$，原式為 $\mathbf{x}' = \mathbf{A}\mathbf{x} \cdots\cdots\cdots \text{(i)}$

$$\begin{vmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda - 35 = 0 \Rightarrow (\lambda+5)(\lambda-7) = 0 \Rightarrow \lambda = -5, 7$$

$$\lambda = -5 \Rightarrow \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix} \mathbf{x}_1 = 0 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ \lambda = 7 \Rightarrow \begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix} \mathbf{x}_2 = 0 \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

得 $\mathbf{D} = \begin{bmatrix} -5 & 0 \\ 0 & 7 \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \Rightarrow \mathbf{S}^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$

令 $\mathbf{x} = \mathbf{S}\mathbf{y}$, (i) $\Rightarrow \mathbf{S}\mathbf{y}' = \mathbf{A}\mathbf{S}\mathbf{y} \Rightarrow \mathbf{y}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{y} \Rightarrow \mathbf{y}' = \mathbf{D}\mathbf{y}$

$$\mathbf{y} = e^{\mathbf{D}t}\mathbf{C} \Rightarrow \mathbf{x} = \mathbf{S}e^{\mathbf{D}t}\mathbf{C} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{7t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2C_1 e^{-5t} + 2C_2 e^{7t} \\ -C_1 e^{-5t} + C_2 e^{7t} \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2C_1 + 2C_2 \\ -C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

解為 $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -e^{-5t} + e^{7t} \\ \frac{1}{2}e^{-5t} + \frac{1}{2}e^{7t} \end{bmatrix}$