(a)Find the Fourier series of the function $f(x)=\dfrac{x^2}{4}$, $-\pi \le x \le \pi$; $f(x+2\pi)=f(x)$. (b)Calculate $\sum_{n=1}^{\infty}\dfrac{1}{n^2}$.

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[解](a) $f(x)$為偶函數 ⇒ 令 $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}a_n \cos nx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x^2}{4}dx=\frac{1}{4\pi}\cdot\frac{x^3}{3}\bigg|_{-\pi}^{\pi}=\frac{2\pi^3}{12\pi}=\frac{\pi^2}{6}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x^2}{4}\cos nxdx=\frac{1}{4n\pi}(x^2\sin nx\Big|_{-\pi}^{\pi}-2\int_{-\pi}^{\pi}x\sin nxdx)$$

$$=\frac{1}{2n^2\pi}(x\cos nx\Big|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\cos nxdx)=\frac{\pi\cos n\pi-(-\pi)\cos(-n\pi)}{2n^2\pi}=\frac{\cos n\pi}{n^2}=\frac{(-1)^n}{n^2}$$

$$f(x)=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

(b)令 $x=\pi$ 代入(a)得

$$\frac{f(\pi^-)+f(\pi^+)}{2}=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi \Rightarrow \frac{\frac{\pi^2}{4}+\frac{\pi^2}{4}}{2}=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{4}=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$