Expand $f(x)=x^2$ for $0<x<L$, in a sine series. [100清大動機7(a)]

[解] $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^L x^2\sin\frac{n\pi x}{L}dx=-\frac{2}{L}\cdot\frac{L}{n\pi}\left(x^2\cos\frac{n\pi x}{L}\Big|_0^L-2\int_0^L x\cos\frac{n\pi x}{L}dx\right)$$

$$=-\frac{2}{n\pi}\left[L^2\cos n\pi-2\cdot\frac{L}{n\pi}\left(x\sin\frac{n\pi x}{L}\Big|_0^L-\int_0^L\sin\frac{n\pi x}{L}dx\right)\right]$$

$$=-\frac{2}{n\pi}\left[L^2(-1)^n+\frac{2L}{n\pi}\int_0^L\sin\frac{n\pi x}{L}dx\right]=-\frac{2}{n\pi}\left[L^2(-1)^n-\frac{2L}{n\pi}\cdot\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L\right]$$

$$=-\frac{2}{n\pi}\left[L^2(-1)^n-\frac{2L^2}{n^2\pi^2}(\cos n\pi-1)\right]=-\frac{2}{n\pi}\left\{L^2(-1)^n-\frac{2L^2}{n^2\pi^2}[(-1)^n-1]\right\}$$

$$f(x)=-\frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{L^2(-1)^n}{n}-\frac{2L^2}{n^3\pi^2}[(-1)^n-1]\right\}\sin\frac{n\pi x}{L},\quad 0<x<L$$