Let $\mathbf{A}$ be a 4×4 matrix. If $\operatorname{adj}\mathbf{A}=\begin{bmatrix}2&0&0&0\\0&2&1&0\\0&4&3&2\\0&-2&-1&2\end{bmatrix}$, (a)calculate the value of det(adj $\mathbf{A}$). What should the value of det($\mathbf{A}$) be? (b)find $\mathbf{A}$. [99 高師大光電 4]

[解](a) $\det(\operatorname{adj}\mathbf{A})=2\begin{vmatrix}2&1&0\\4&3&2\\-2&-1&2\end{vmatrix}\xrightarrow{R_{23}(-1)}=2\begin{vmatrix}2&1&0\\4&3&2\\-6&-4&0\end{vmatrix}=2(-2)\begin{vmatrix}2&1\\-6&-4\end{vmatrix}=(-4)(-8+6)=8$

$$\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}\Rightarrow\det(\mathbf{A})\cdot\det(\mathbf{A}^{-1})=1\cdots\cdots\cdots(\text{i})$$

而 $\mathbf{A}^{-1}=\dfrac{1}{\det(\mathbf{A})}\operatorname{adj}\mathbf{A}\Rightarrow\det(\mathbf{A}^{-1})=\dfrac{1}{[\det(\mathbf{A})]^4}\cdot\det(\operatorname{adj}\mathbf{A})$

代入(i) $\Rightarrow\det(\mathbf{A})\cdot\dfrac{1}{[\det(\mathbf{A})]^4}\cdot\det(\operatorname{adj}\mathbf{A})=1\Rightarrow\dfrac{1}{[\det(\mathbf{A})]^3}\cdot 8=1\Rightarrow\det(\mathbf{A})=2$

(b) $\mathbf{A}^{-1}=\dfrac{1}{\det(\mathbf{A})}\operatorname{adj}\mathbf{A}=\dfrac{1}{2}\begin{bmatrix}2&0&0&0\\0&2&1&0\\0&4&3&2\\0&-2&-1&2\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&\frac{1}{2}&0\\0&2&\frac{3}{2}&1\\0&-1&-\frac{1}{2}&1\end{bmatrix}$

$$\mathbf{A}=\frac{1}{\det(\mathbf{A}^{-1})}\operatorname{adj}\mathbf{A}^{-1}=2\begin{bmatrix}\frac{1}{2}&0&0&0\\0&2&-\frac{1}{2}&\frac{1}{2}\\0&-3&1&-1\\0&\frac{1}{2}&0&\frac{1}{2}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&4&-1&1\\0&-6&2&-2\\0&1&0&1\end{bmatrix}$$