Find the value of $k$ which makes the following system $\mathbf{Ax}=\mathbf{b}$ consistent; also find the determinant of matrix $\mathbf{A}$.

$$\mathbf{A}=\begin{bmatrix}1&2&3&1\\3&2&1&4\\2&6&10&3\\1&1&1&1\end{bmatrix},\ \mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}3\\6\\6\\k\end{bmatrix}.$$ [99 交大機械甲 2]

[解]增廣矩陣 $\begin{bmatrix}1&2&3&1&3\\3&2&1&4&6\\2&6&10&3&6\\1&1&1&1&k\end{bmatrix}\xrightarrow{R_{12}(-3);\,R_{13}(-2);\,R_{14}(-1)}\begin{bmatrix}1&2&3&1&3\\0&-4&-8&1&-3\\0&2&4&1&0\\0&-1&-2&0&k-3\end{bmatrix}$

$\xrightarrow{R_{24}}\begin{bmatrix}1&2&3&1&3\\0&-1&-2&0&k-3\\0&2&4&1&0\\0&-4&-8&1&-3\end{bmatrix}\xrightarrow{R_{23}(2);\,R_{24}(-4)}\begin{bmatrix}1&2&3&1&3\\0&-1&-2&0&k-3\\0&0&0&1&2k-6\\0&0&0&1&-4k+9\end{bmatrix}$

$\xrightarrow{R_{34}(-1)}\begin{bmatrix}1&2&3&1&3\\0&-1&-2&0&k-3\\0&0&0&1&2k-6\\0&0&0&0&-6k+15\end{bmatrix}$

增廣矩陣與係數矩陣的$\text{rank}=3\Rightarrow -6k+15=0\Rightarrow k=\dfrac{5}{2}$

係數矩陣的rank小於$4\Rightarrow|\mathbf{A}|=0$