With a matrix **A** and a vector **b** given: $\mathbf{A}=\begin{bmatrix}1&3&5&4\\-2&2&6&2\\3&1&-1&3\end{bmatrix}$ $\mathbf{b}=\begin{bmatrix}1\\2\\3\end{bmatrix}$.

(a) Find the rank of **A**.

(b) Find the homogeneous solution to $\mathbf{Ax}=0$.

(c) Solve $\mathbf{Ax}=\mathbf{b}$ and explain the relation between **x** and the homogeneous solution of $\mathbf{Ax}=0$. [98 中山機電 III 1]

[解](a) $\begin{bmatrix}1&3&5&4\\-2&2&6&2\\3&1&-1&3\end{bmatrix}\xrightarrow{R_{12}(2);\,R_{13}(-3)}\begin{bmatrix}1&3&5&4\\0&8&16&10\\0&-8&-16&-9\end{bmatrix}$

$\xrightarrow{R_{23}(1)}\begin{bmatrix}1&3&5&4\\0&8&16&10\\0&0&0&1\end{bmatrix}\Rightarrow$ 矩陣 **A** 的秩為 3

(b)方程式 $\mathbf{Ax}=0$ 化簡為 $\begin{cases}x_1+3x_2+5x_3+4x_4=0\\8x_2+16x_3+10x_4=0\\x_4=0\end{cases}$

令 $x_3=C\Rightarrow x_2=-2C,\ x_1=C$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=C\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}\Rightarrow$ 解為 $\mathbf{x}=\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}$

(c) $\begin{bmatrix}1&3&5&4&1\\-2&2&6&2&2\\3&1&-1&3&3\end{bmatrix}\xrightarrow{R_{12}(2);\,R_{13}(-3)}\begin{bmatrix}1&3&5&4&1\\0&8&16&10&4\\0&-8&-16&-9&0\end{bmatrix}$

$\xrightarrow{R_{23}(1)}\begin{bmatrix}1&3&5&4&1\\0&8&16&10&4\\0&0&0&1&4\end{bmatrix}\Rightarrow\begin{cases}x_1+3x_2+5x_3+4x_4=1\\8x_2+16x_3+10x_4=4\\x_4=4\end{cases}$

$\begin{cases}x_1+3x_2+5x_3=-15\\2x_2+4x_3=-9\end{cases}$, 令 $x_3=C\Rightarrow x_2=-2C-\frac{9}{2},\ x_1=C-\frac{3}{2}$

$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}C-\frac{3}{2}\\-2C-\frac{9}{2}\\C\\4\end{bmatrix}=C\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}+\begin{bmatrix}-\frac{3}{2}\\-\frac{9}{2}\\0\\4\end{bmatrix}=$ 齊性解 + 特解