It is known that $\mathbf{A}=\begin{bmatrix}2 & 1 & -1\\ 3 & 2 & -3\\ 3 & 1 & -2\end{bmatrix}$. If $\mathbf{B}=\mathbf{A}^6-2\mathbf{A}^5-3\mathbf{A}^4+9\mathbf{A}^3-4\mathbf{A}^2-6\mathbf{A}+8\mathbf{I}$ find $\mathbf{B}$ and $e^{\mathbf{B}}$, both solution should be expressed in terms of a 3×3 matrix. [97 台大土木 1]

[解] $|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}2-\lambda & 1 & -1\\ 3 & 2-\lambda & -3\\ 3 & 1 & -2-\lambda\end{vmatrix}=0\Rightarrow-(\lambda+2)(\lambda-2)^2-12-6(\lambda-2)+3(\lambda+2)=0$

$\lambda^3-2\lambda^2-\lambda+2=0\Rightarrow(\lambda+1)(\lambda-1)(\lambda-2)=0\Rightarrow\lambda=-1,1,2$

$\lambda=-1,\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\Rightarrow\begin{bmatrix}3 & 1 & -1\\ 3 & 3 & -3\\ 3 & 1 & -1\end{bmatrix}\mathbf{x}=0\Rightarrow\mathbf{x}_1=\begin{bmatrix}0\\1\\1\end{bmatrix}$

$\lambda=1,\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\Rightarrow\begin{bmatrix}1 & 1 & -1\\ 3 & 1 & -3\\ 3 & 1 & -3\end{bmatrix}\mathbf{x}=0\Rightarrow\mathbf{x}_2=\begin{bmatrix}1\\0\\1\end{bmatrix}$

$\lambda=2,\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\Rightarrow\begin{bmatrix}0 & 1 & -1\\ 3 & 0 & -3\\ 3 & 1 & -4\end{bmatrix}\mathbf{x}=0\Rightarrow\mathbf{x}_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$

由Cayley–Hamilton定理知 $\mathbf{A}^3-2\mathbf{A}^2-\mathbf{A}+2\mathbf{I}=0$

$\mathbf{B}=\mathbf{A}^6-2\mathbf{A}^5-3\mathbf{A}^4+9\mathbf{A}^3-4\mathbf{A}^2-6\mathbf{A}+8\mathbf{I}=(\mathbf{A}^3-2\mathbf{A}^2-\mathbf{A}+2\mathbf{I})(\mathbf{A}^3-2\mathbf{A}+3\mathbf{I})+\mathbf{A}+2\mathbf{I}$

$=\mathbf{A}+2\mathbf{I}=\begin{bmatrix}2 & 1 & -1\\ 3 & 2 & -3\\ 3 & 1 & -2\end{bmatrix}+2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}4 & 1 & -1\\ 3 & 4 & -3\\ 3 & 1 & 0\end{bmatrix}$

且$\mathbf{B}$的特徵值為$\mathbf{A}$的特徵值加2得1, 3, 4，$\mathbf{B}$與$\mathbf{A}$有相同的特徵向量

$\mathbf{D}=\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{bmatrix},\ \mathbf{S}=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}\Rightarrow\mathbf{S}^{-1}=\begin{bmatrix}-1 & 0 & 1\\ 0 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$

$e^{\mathbf{B}}=\mathbf{S}e^{\mathbf{D}}\mathbf{S}^{-1}=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}e & 0 & 0\\ 0 & e^3 & 0\\ 0 & 0 & e^4\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\ 0 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}$

$=\begin{bmatrix}0 & e^3 & e^4\\ e & 0 & e^4\\ e & e^3 & e^4\end{bmatrix}\begin{bmatrix}-1 & 1 & -1\\ 1 & -1 & -2\\ -1 & -2 & 2\end{bmatrix}=\begin{bmatrix}e^3-e^4 & -e^3-2e^4 & -2e^3+2e^4\\ -e-e^4 & e-2e^4 & -e+2e^4\\ -e+e^3-e^4 & e-e^3-2e^4 & -e-2e^3+2e^4\end{bmatrix}$