Find out what type of conic section is represented by given quadratic form. $Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = 128$.Transform it to principal axes. [97 中央機械丁光電甲 5]

[解]$(-30)^2 - 4\cdot 17\cdot 17 = -256 < 0 \Rightarrow$ elliptic type

$$Q = 17x_1^2 - 30x_1x_2 + 17x_2^2 = [x_1 \quad x_2]\begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = 1$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}$

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 17-\lambda & -15 \\ -15 & 17-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 2)(\lambda - 32) = 0 \Rightarrow \lambda = 2, 32$$

$$\lambda = 2, (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} 15 & -15 \\ -15 & 15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \mathbf{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda = 32, (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} -15 & -15 \\ -15 & -15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \mathbf{e}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Let $\mathbf{x} = \mathbf{P}\mathbf{y}$, where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, then

$$Q = \mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{P}\mathbf{y})^T\mathbf{A}(\mathbf{P}\mathbf{y}) = \mathbf{y}^T\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{y}$$

$$= [y_1 \quad y_2]\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 17 & -15 \\ -15 & 17 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [y_1 \quad y_2]\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -16\sqrt{2} & 16\sqrt{2} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= [y_1 \quad y_2]\begin{bmatrix} 2 & 0 \\ 0 & 32 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2y_1^2 + 32y_2^2$$