Solve the system of linear ODEs by matrix diagonalization (similarity transform of eigen-matrix) $\begin{cases} y_1' = 3y_1 + 3y_2 + 8 \\ y_2' = y_1 + 5y_2 + 4e^{3t} \end{cases}$. [106 成大機械 2]

[解]令 $\mathbf{x} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}, \mathbf{h} = \begin{bmatrix} 8 \\ 4e^{3t} \end{bmatrix}$，原式為 $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{h}$

$$\begin{vmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 8\lambda + 12 = 0 \Rightarrow \lambda = 2, 6$$

$\lambda = 2$時, $\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}\mathbf{x}_1 = 0 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$，$\lambda = 6$時, $\begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}\mathbf{x}_2 = 0 \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

得 $\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}, \mathbf{S} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \mathbf{S}^{-1} = \dfrac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

令 $\mathbf{x} = \mathbf{S}\mathbf{y} \Rightarrow \mathbf{x}' = \mathbf{S}\mathbf{y}' \Rightarrow \mathbf{y}' = \mathbf{S}^{-1}\mathbf{x}' \Rightarrow \mathbf{y}' = \mathbf{S}^{-1}(\mathbf{A}\mathbf{x} + \mathbf{h}) \Rightarrow \dot{\mathbf{y}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{y} + \mathbf{S}^{-1}\mathbf{h} = \mathbf{D}\mathbf{y} + \mathbf{S}^{-1}\mathbf{h}$

$$\mathbf{y} = e^{\mathbf{D}t}\left[\int e^{-\mathbf{D}t}\mathbf{S}^{-1}\mathbf{h}\,dt + \mathbf{c}\right] = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{6t} \end{bmatrix}\left[\int \frac{1}{4}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-6t} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 4e^{3t} \end{bmatrix}dt + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right]$$

$$= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{6t} \end{bmatrix}\left[\int \frac{1}{4}\begin{bmatrix} 8e^{-2t} - 4e^{t} \\ 8e^{-6t} + 12e^{-3t} \end{bmatrix}dt + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right] = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{6t} \end{bmatrix}\left[\begin{bmatrix} -e^{-2t} - e^{t} \\ -\frac{1}{3}e^{-6t} - e^{-3t} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right]$$

$$= \begin{bmatrix} -1 - e^{3t} + c_1 e^{2t} \\ -\frac{1}{3} - e^{3t} + c_2 e^{6t} \end{bmatrix}$$

$$\mathbf{x} = \mathbf{S}\mathbf{y} = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 - e^{3t} + c_1 e^{2t} \\ -\frac{1}{3} - e^{3t} + c_2 e^{6t} \end{bmatrix} = \begin{bmatrix} 3c_1 e^{2t} + c_2 e^{6t} - \frac{10}{3} - 4e^{3t} \\ -c_1 e^{2t} + c_2 e^{6t} + \frac{2}{3} \end{bmatrix}$$