Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. [104 聯合環安 6]

[解] $|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$-\lambda(\lambda-1)(\lambda+2) + 12 + 12 + 3(\lambda-1) + 12(\lambda+2) + 4\lambda = 0$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \Rightarrow (\lambda-5)(\lambda+3)^2 \Rightarrow \lambda = 5, -3, -3$

$\lambda = 5, (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$x_1 : x_2 : x_3 = \begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix} : \begin{vmatrix} -3 & -7 \\ -6 & 2 \end{vmatrix} : \begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix} = 1 : 2 : (-1) \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\lambda = -3, (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow x_1 + 2x_2 - 3x_3 = 0$

$\Leftrightarrow x_3 = C_1, x_2 = C_2 \Rightarrow x_1 = 3C_1 - 2C_2 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3C_1 - 2C_2 \\ C_2 \\ C_1 \end{bmatrix} = C_1\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\mathbf{x}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$