If $\begin{cases} y_1' = -y_1 + y_2 \\ y_2' = y_1 - 2y_2 + y_3 \\ y_3' = y_2 - y_3 \end{cases}$, $y_1(0)=2$, $y_2(0)=3$, $y_3(0)=0$, solve $y_1(t)$, $y_2(t)$, $y_3(t)$. [104 海洋造船 3]

[解]令$\mathbf{x} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$，原式為$\mathbf{x}' = \mathbf{A}\mathbf{x} \cdots\cdots\cdots(\text{i})$

$$\begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow -(\lambda+1)^2(\lambda+2) + (\lambda+1) + (\lambda+1) = 0$$

$$(\lambda+1)[(\lambda^2+3\lambda+2)-2] = 0 \Rightarrow \lambda(\lambda+1)(\lambda+3) = 0 \Rightarrow \lambda = 0, -1, -3$$

$$\lambda = 0 \Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\mathbf{x}_1 = 0 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \lambda = -1 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x}_2 = 0 \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda = -3 \Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\mathbf{x}_3 = 0 \Rightarrow \mathbf{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

得 $\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow \mathbf{S}^{-1} = -\frac{1}{6}\begin{bmatrix} -2 & -2 & -2 \\ -3 & 0 & 3 \\ -1 & 2 & -1 \end{bmatrix}$



令 $\mathbf{x} = \mathbf{S}\mathbf{y}$, (i) $\Rightarrow \mathbf{S}\mathbf{y}' = \mathbf{A}\mathbf{S}\mathbf{y} \Rightarrow \mathbf{y}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{y} \Rightarrow \mathbf{y}' = \mathbf{D}\mathbf{y}$

$$\mathbf{y} = e^{\mathbf{D}t}\mathbf{C} \Rightarrow \mathbf{x} = \mathbf{S}e^{\mathbf{D}t}\mathbf{C} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} C_1 + C_2e^{-t} + C_3e^{-3t} \\ C_1 - 2C_3e^{-3t} \\ C_1 - C_2e^{-t} + C_3e^{-3t} \end{bmatrix}$$

$$\begin{bmatrix} y_1(0) = 2 \\ y_2(0) = 3 \\ y_3(0) = 0 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 + C_2 + C_3 \\ C_1 - 2C_3 \\ C_1 - C_2 + C_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ 1 \\ -\frac{2}{3} \end{bmatrix}$$

解為$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} \frac{5}{3} + e^{-t} - \frac{2}{3}e^{-3t} \\ \frac{5}{3} + \frac{4}{3}e^{-3t} \\ \frac{5}{3} - e^{-t} - \frac{2}{3}e^{-3t} \end{bmatrix}$