Define the matrices $\mathbf{A}$ and $\mathbf{\Lambda}$ as $\mathbf{A}=\begin{bmatrix}-3 & 0 & 0\\ 1 & -2 & 0\\ 0 & 0 & -4\end{bmatrix}$, and $\Lambda=\begin{bmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}$, respectively.

(a)If $\mathbf{\Lambda}$ is the diagonal matrix of $\mathbf{A}$, find the value of a, b, and c for $a\le b\le c$.

(b)Find a matrix $\mathbf{X}$ such $\mathbf{\Lambda}=\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$.

(c)Find a matrix $\mathbf{B}$ such that $\mathbf{B}^3=\mathbf{A}$. [104 彰師大物理甲光電甲 4]

[解] $|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}-3-\lambda & 0 & 0\\ 1 & -2-\lambda & 0\\ 0 & 0 & -4-\lambda\end{vmatrix}=0\Rightarrow-(\lambda+4)(\lambda+3)(\lambda+2)=0\Rightarrow\lambda=-4,-3,-2$

得 $a=-4, b=-3, c=-2$

$\lambda=-4,(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\Rightarrow\begin{bmatrix}1 & 0 & 0\\ 1 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}\mathbf{x}=0\Rightarrow\mathbf{x}_1=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$

$\lambda=-3,(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\Rightarrow\begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\mathbf{x}=0\Rightarrow\mathbf{x}_2=\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$

$\lambda=-2,(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=0\Rightarrow\begin{bmatrix}-1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 2\end{bmatrix}\mathbf{x}=0\Rightarrow\mathbf{x}_3=\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$

得 $\Lambda=\begin{bmatrix}-4 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -2\end{bmatrix}$, $\mathbf{X}=\begin{bmatrix}0 & -1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 0\end{bmatrix}\Rightarrow\mathbf{X}^{-1}=\begin{bmatrix}0 & 0 & 1\\ -1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}$

$$\mathbf{B}=\sqrt[3]{\mathbf{A}}=\mathbf{X}\sqrt[3]{\Lambda}\mathbf{X}^{-1}=\begin{bmatrix}0 & -1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\sqrt[3]{-4} & 0 & 0\\ 0 & \sqrt[3]{-3} & 0\\ 0 & 0 & \sqrt[3]{-2}\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ -1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}0 & -\sqrt[3]{-3} & 0\\ 0 & \sqrt[3]{-3} & \sqrt[3]{-2}\\ \sqrt[3]{-4} & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ -1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}=\begin{bmatrix}\sqrt[3]{-3} & 0 & 0\\ -\sqrt[3]{-3}+\sqrt[3]{-2} & \sqrt[3]{-2} & 0\\ 0 & 0 & \sqrt[3]{-4}\end{bmatrix}$$