

(1) Find eigenvalues and orthogonal eigenvectors of matrix $\mathbf{A} = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix}$.

(2) Construct an orthogonal matrix \mathbf{P} from the eigenvectors. What is \mathbf{P} and \mathbf{P}^{-1} ? [104 中興機械 4]

[解] $\begin{vmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix} = 0 \Rightarrow -(\lambda-5)(\lambda-6)(\lambda-7) + 4(\lambda-7) + 4(\lambda-5) = 0$

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0 \Rightarrow (\lambda-3)(\lambda-6)(\lambda-9) \Rightarrow \lambda = 3, 6, 9$$

$$\lambda = 3, \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow x_1 : x_2 : x_3 = \begin{vmatrix} -2 & 0 \\ 3 & 2 \end{vmatrix} : \begin{vmatrix} 0 & 4 \\ 2 & -2 \end{vmatrix} : \begin{vmatrix} 4 & -2 \\ -2 & 3 \end{vmatrix} = 1 : 2 : (-2)$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \Rightarrow \mathbf{e}_1 = \frac{\mathbf{x}_1}{|\mathbf{x}_1|} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

同理 $\lambda = 6, \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \Rightarrow \mathbf{e}_2 = \frac{\mathbf{x}_2}{|\mathbf{x}_2|} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$

$$\lambda = 9, \begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \mathbf{x}_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{e}_3 = \frac{\mathbf{x}_3}{|\mathbf{x}_3|} = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$