A matrix $\mathbf{A}=\begin{bmatrix}-1 & 2 & a\\ 2 & 4 & b\\ 2 & 1 & c\end{bmatrix}$ has two eigenvectors $\boldsymbol{\varphi}_1=\begin{bmatrix}0\\1\\1\end{bmatrix}$, $\boldsymbol{\varphi}_2=\begin{bmatrix}-1\\0\\2\end{bmatrix}$, where a, b, c are unknown constants. Find (1)the values of a, b, c, (2)the three eigenvalues λ_1, λ_2, λ_3, and (3)the third eigenvectors φ_3. [104 中興土木甲 2]

[解] $\mathbf{A}\boldsymbol{\varphi}_1-\lambda_1\boldsymbol{\varphi}_1=0\Rightarrow\begin{bmatrix}-1 & 2 & a\\ 2 & 4 & b\\ 2 & 1 & c\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}-\lambda_1\begin{bmatrix}0\\1\\1\end{bmatrix}=0\Rightarrow\begin{bmatrix}2+a\\4+b-\lambda_1\\1+c-\lambda_1\end{bmatrix}=0$

$$\mathbf{A}\boldsymbol{\varphi}_2-\lambda_2\boldsymbol{\varphi}_2=0\Rightarrow\begin{bmatrix}-1 & 2 & a\\ 2 & 4 & b\\ 2 & 1 & c\end{bmatrix}\begin{bmatrix}-1\\0\\2\end{bmatrix}-\lambda_2\begin{bmatrix}-1\\0\\2\end{bmatrix}=0\Rightarrow\begin{bmatrix}1+2a+\lambda_2\\-2+2b\\-2+2c-2\lambda_2\end{bmatrix}=0$$

得 $a=-2, b=1, \lambda_1=5, \lambda_2=3, c=4$

$$|\mathbf{A}-\lambda\mathbf{I}|=0\Rightarrow\begin{vmatrix}-1-\lambda & 2 & -2\\ 2 & 4-\lambda & 1\\ 2 & 1 & 4-\lambda\end{vmatrix}=0\Rightarrow-(\lambda+1)(\lambda-4)^2+(\lambda+1)=0$$

$$(\lambda+1)(\lambda-3)(\lambda-5)=0\Rightarrow\lambda_1=5, \lambda_2=3, \lambda_3=-1$$

$$\lambda=-1, (\mathbf{A}-\lambda\mathbf{I})\boldsymbol{\varphi}=0\Rightarrow\begin{bmatrix}0 & 2 & -2\\ 2 & 5 & 1\\ 2 & 1 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=0\Rightarrow\boldsymbol{\varphi}_3=\begin{bmatrix}-3\\1\\1\end{bmatrix}$$