

Find the eigenvalues and corresponding eigenvectors of matrix $\mathbf{A} = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. [104 中正資工 4]

$$[\text{解}] |\mathbf{A} - \lambda \mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$-(\lambda-5)^2(\lambda-2) + 16 + 16 + 4(\lambda-5) + 4(\lambda-5) + 16(\lambda-2) = 0 \Rightarrow \lambda^3 - 12\lambda^2 + 21\lambda - 10 = 0$$

$$(\lambda-10)(\lambda-1)^2 = 0 \Rightarrow \lambda = 10, 1, 1$$

$$\lambda = 10 \text{ 時, } (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -3 \end{bmatrix} \mathbf{x} = 0$$

$$x_1 : x_2 : x_3 = \begin{vmatrix} 4 & 2 \\ -5 & 2 \end{vmatrix} : \begin{vmatrix} 2 & -5 \\ 2 & 4 \end{vmatrix} : \begin{vmatrix} -5 & 4 \\ 4 & -5 \end{vmatrix} = 18 : 18 : 9 = 2 : 2 : 1 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda = 1 \text{ 時, } (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \mathbf{x} = 0 \Rightarrow 2x_1 + 2x_2 + x_3 = 0 \Rightarrow \text{令 } x_1 = C_1, x_2 = C_2$$

$$\text{則 } x_3 = -2C_1 - 2C_2 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$