

Let $\mathbf{A} = \begin{bmatrix} 5 & 10 & -10 \\ 10 & 5 & -20 \\ 5 & -5 & -10 \end{bmatrix}$, (a) Find the eigenvalues and eigenvectors of \mathbf{A} . (b) Find a square

matrix \mathbf{X} so that $\mathbf{X}^{-1}\mathbf{A}\mathbf{X}$ is a diagonal matrix. (c) Evaluate $e^{\mathbf{A}t}$. [104 中正機械 3]

$$\text{[解](a) } |\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 10 & -10 \\ 10 & 5-\lambda & -20 \\ 5 & -5 & -10-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda+15)(\lambda-15) = 0, \lambda = 0, -15, 15$$

$$\lambda = 0, (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} 5 & 10 & -10 \\ 10 & 5 & -20 \\ 5 & -5 & -10 \end{bmatrix} \mathbf{x} = 0 \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = -15, (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} 20 & 10 & -10 \\ 10 & 20 & -20 \\ 5 & -5 & 5 \end{bmatrix} \mathbf{x} = 0 \Rightarrow \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = 15, (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} -10 & 10 & -10 \\ 10 & -10 & -20 \\ 5 & -5 & 5 \end{bmatrix} \mathbf{x} = 0 \Rightarrow \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{(b) } \mathbf{X} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow \mathbf{X}^{-1} = \frac{1}{-3} \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -2 \\ -1 & -2 & 2 \end{bmatrix}$$

$$\text{(c) } e^{\mathbf{A}t} = \mathbf{X}e^{\mathbf{A}t}\mathbf{X}^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-15t} & 0 \\ 0 & 0 & e^{15t} \end{bmatrix} \frac{1}{-3} \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -2 \\ -1 & -2 & 2 \end{bmatrix}$$

$$= \frac{1}{-3} \begin{bmatrix} 2 & 0 & e^{15t} \\ 0 & e^{-15t} & e^{15t} \\ 1 & e^{-15t} & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -2 \\ -1 & -2 & 2 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} -2 - e^{15t} & 2 - 2e^{15t} & -2 + 2e^{15t} \\ e^{-15t} - e^{15t} & -e^{-15t} - 2e^{15t} & -2e^{-15t} + 2e^{15t} \\ -1 + e^{-15t} & 1 - e^{-15t} & -1 - 2e^{-15t} \end{bmatrix}$$