Find the value of the determinant $\begin{vmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ 1 & \beta & \beta^2 & \beta^3 \\ 1 & \gamma & \gamma^2 & \gamma^3 \\ 1 & \delta & \delta^2 & \delta^3 \end{vmatrix}$. [103 台大工科海洋 1]

[解] $\begin{vmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ 0 & \beta-\alpha & \beta^2-\alpha^2 & \beta^3-\alpha^3 \\ 0 & \gamma-\alpha & \gamma^2-\alpha^2 & \gamma^3-\alpha^3 \\ 0 & \delta-\alpha & \delta^2-\alpha^2 & \delta^3-\alpha^3 \end{vmatrix} = \begin{vmatrix} \beta-\alpha & \beta^2-\alpha^2 & \beta^3-\alpha^3 \\ \gamma-\alpha & \gamma^2-\alpha^2 & \gamma^3-\alpha^3 \\ \delta-\alpha & \delta^2-\alpha^2 & \delta^3-\alpha^3 \end{vmatrix}$

$= (\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\begin{vmatrix} 1 & \beta+\alpha & \beta^2+\beta\alpha+\alpha^2 \\ 1 & \gamma+\alpha & \gamma^2+\gamma\alpha+\alpha^2 \\ 1 & \delta+\alpha & \delta^2+\delta\alpha+\alpha^2 \end{vmatrix}$

$= (\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\begin{vmatrix} 1 & \beta+\alpha & \beta^2+\beta\alpha+\alpha^2 \\ 0 & \gamma-\beta & \gamma^2+\gamma\alpha-\beta^2-\beta\alpha \\ 0 & \delta-\beta & \delta^2+\delta\alpha-\beta^2-\beta\alpha \end{vmatrix}$

$= (\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\begin{vmatrix} \gamma-\beta & \gamma^2+\gamma\alpha-\beta^2-\beta\alpha \\ \delta-\beta & \delta^2+\delta\alpha-\beta^2-\beta\alpha \end{vmatrix}$

$= (\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\begin{vmatrix} \gamma-\beta & (\gamma-\beta)(\gamma+\beta+\alpha) \\ \delta-\beta & (\delta-\beta)(\delta+\beta+\alpha) \end{vmatrix}$

$= (\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)(\gamma-\beta)(\delta-\beta)\begin{vmatrix} 1 & \gamma+\beta+\alpha \\ 1 & \delta+\beta+\alpha \end{vmatrix} = (\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)(\gamma-\beta)(\delta-\beta)(\delta-\gamma)$