Solve the nonhomogeneous ODE $y'' + y = \sec x = \dfrac{1}{\cos x}$. [99 海洋河海 4]

[解]$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i \Rightarrow y_h(x) = C_1\cos x + C_2\sin x$

令 $y_p(x) = u_1\cos x + u_2\sin x \Rightarrow y_p' = (-u_1\sin x + u_2\cos x) + (u_1'\cos x + u_2'\sin x)$

令 $u_1'\cos x + u_2'\sin x = 0\cdots\cdots\cdots\cdots$(i)

$y_p'' = (-u_1\cos x - u_2\sin x) + (-u_1'\sin x + u_2'\cos x)$，代入原式

$[(-u_1\cos x - u_2\sin x) + (-u_1'\sin x + u_2'\cos x)] + (u_1\cos x + u_2\sin x) = \tan x$

$-u_1'\sin x + u_2'\cos x = \sec x\cdots\cdots\cdots\cdots$(ii)

由(i)與(ii)得

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{W(\cos x, \sin x)} = -\sin x \sec x = -\tan x$$

$u_1 = \int -\tan x\,dx = -\ln|\sec x|$

$$u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{W(\cos x, \sin x)} = 1 \Rightarrow u_2 = \int 1\,dx = x$$

$y(x) = y_h(x) + y_p(x) = C_1\cos x + C_2\sin x - (\ln|\sec x|)\cos x + (x)\sin x$

$\qquad = (C_1 - \ln|\sec x|)\cos x + (C_2 + x)\sin x$