Solve the problem $x\dfrac{dy}{dx} = y - \sqrt{y^2 + x^2}$. [94 高應大機械 1(1)]

[解]令 $y = ux \Rightarrow y' = xu' + u$

原式 $\Rightarrow x(xu' + u) = ux - \sqrt{u^2x^2 + x^2} \Rightarrow xu' + u = u - \sqrt{u^2 + 1} \Rightarrow xu' + \sqrt{u^2 + 1} = 0$

$x\dfrac{du}{dx} + \sqrt{u^2 + 1} = 0 \Rightarrow \int \dfrac{du}{\sqrt{u^2 + 1}} + \int \dfrac{dx}{x} = C \cdots\cdots\cdots \text{(i)}$

令 $u = \tan\theta \Rightarrow du = \sec^2\theta d\theta, \sqrt{u^2 + 1} = \sec\theta$

$\text{(i)} \Rightarrow \int \dfrac{\sec^2\theta d\theta}{\sec\theta} + \int \dfrac{dx}{x} = C \Rightarrow \int \sec\theta d\theta + \int \dfrac{dx}{x} = C \Rightarrow \ln(\sec\theta + \tan\theta) + \ln x = k$

$\ln(\sqrt{u^2 + 1} + u) + \ln x = k \Rightarrow \ln[x(\sqrt{u^2 + 1} + u)] = k \Rightarrow x(\sqrt{u^2 + 1} + u) = C$

$x[\sqrt{(\dfrac{y}{x})^2 + 1} + \dfrac{y}{x}] = C \Rightarrow \sqrt{y^2 + x^2} + y = C$