Find the general solution to $2y'' + xy' + y = 0$ in the form of power series about the ordinary point $x = 0$. [94 中興材料 6]

[解]設 $y = \sum_{m=0}^{\infty} A_m x^m \Rightarrow y' = \sum_{m=0}^{\infty} m A_m x^{m-1} \Rightarrow y'' = \sum_{m=0}^{\infty} m(m-1) A_m x^{m-2}$

原式 $\Rightarrow 2\sum_{m=0}^{\infty} m(m-1)A_m x^{m-2} + \sum_{m=0}^{\infty} mA_m x^m + \sum_{m=0}^{\infty} A_m x^m = 0$

$$2\sum_{m=2}^{\infty} m(m-1)A_m x^{m-2} + \sum_{m=0}^{\infty} mA_m x^m + \sum_{m=0}^{\infty} A_m x^m = 0$$

$$2\sum_{m=2}^{\infty} m(m-1)A_m x^{m-2} + \sum_{m=2}^{\infty} (m-2)A_{m-2} x^{m-2} + \sum_{m=2}^{\infty} A_{m-2} x^{m-2} = 0$$

$$\sum_{m=2}^{\infty} [2m(m-1)A_m + (m-2+1)A_{m-2}] x^{m-2} = 0$$

$$2m(m-1)A_m + (m-1)A_{m-2} = 0 \Rightarrow 2mA_m + A_{m-2} = 0 \quad m = 2, 3, 4, \cdots$$

$$A_2 = -\frac{A_0}{4},\ A_4 = -\frac{A_2}{8} = \frac{A_0}{4\cdot 8},\ A_6 = -\frac{A_4}{12} = -\frac{A_0}{4\cdot 8\cdot 12}, \cdots$$

$$A_3 = -\frac{A_1}{6},\ A_5 = -\frac{A_3}{10} = \frac{A_1}{6\cdot 10},\ A_7 = -\frac{A_5}{14} = -\frac{A_1}{6\cdot 10\cdot 14}, \cdots$$

$$y = A_0\left(1 - \frac{x^2}{4} + \frac{x^4}{4\cdot 8} - \frac{x^6}{4\cdot 8\cdot 12} + \cdots\right) + A_1\left(x - \frac{x^3}{6} + \frac{x^5}{6\cdot 10} - \frac{x^7}{6\cdot 10\cdot 14} + \cdots\right)$$