

Solve the differential equation  $x^3y'' + 7x^2y' + 9xy = 1$ . [90 清大材料 2]

[解]原式  $\Rightarrow x^2y'' + 7xy' + 9y = \frac{1}{x}$ .....(i)

令  $x = e^t \Rightarrow t = \ln x$

(i)  $\Rightarrow \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = e^{-t}$ .....(ii)

(ii)式的特徵方程式  $\lambda^2 + 6\lambda + 9 = 0 \Rightarrow (\lambda + 3)^2 = 0 \Rightarrow \lambda = -3, -3$

$y_h = (C_1 + C_2t)e^{-3t}$

令  $y_p = Ae^{-t} \Rightarrow y'_p = -Ae^{-t}, y''_p = Ae^{-t}$

代入(ii)式  $\Rightarrow Ae^{-t} - 6Ae^{-t} + 9Ae^{-t} = e^{-t} \Rightarrow A = \frac{1}{4}$

$y = y_h + y_p = (C_1 + C_2t)e^{-3t} + \frac{1}{4}e^{-t} = (C_1 + C_2 \ln x)x^{-3} + \frac{1}{4x}$