

Solve the ordinary differential equation  $y^2 dx + (x^2 - xy - y^2) dy = 0$ . [89 北科大自動化甲 1(a)]

[解] 令  $y = ux \Rightarrow dy = xdu + udx$ ，原式為

$$(ux)^2 dx + [x^2 - x \cdot ux - (ux)^2](xdu + udx) = 0 \Rightarrow u^2 dx + (1 - u - u^2)(xdu + udx) = 0$$

$$[u^2 + (1 - u - u^2)u] dx + (1 - u - u^2)xdu = 0 \Rightarrow (-u^3 + u) dx + (1 - u - u^2)xdu = 0$$

$$(u^3 - u) dx + (u^2 + u - 1)xdu = 0 \Rightarrow u(u+1)(u-1) dx + (u^2 + u - 1)xdu = 0$$

$$\frac{dx}{x} + \frac{u^2 + u - 1}{u(u+1)(u-1)} du = 0 \dots\dots\dots (i)$$

$$\hookrightarrow \frac{u^2 + u - 1}{u(u+1)(u-1)} = \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u-1}$$

$$u^2 + u - 1 = A(u+1)(u-1) + Bu(u-1) + Cu(u+1)$$

$$u = 0 \Rightarrow -1 = -A \Rightarrow A = 1$$

$$u = -1 \Rightarrow -1 = 2B \Rightarrow B = -\frac{1}{2}$$

$$u = 1 \Rightarrow 1 = 2C \Rightarrow C = \frac{1}{2}$$

$$(i) \Rightarrow \frac{dx}{x} + \left[ \frac{1}{u} - \frac{1}{2(u+1)} + \frac{1}{2(u-1)} \right] du = 0 \Rightarrow \int \frac{dx}{x} + \int \left[ \frac{1}{u} - \frac{1}{2(u+1)} + \frac{1}{2(u-1)} \right] du = k$$

$$\ln x + \ln u - \frac{1}{2} \ln(u+1) + \frac{1}{2} \ln(u-1) = k \Rightarrow \ln \frac{xu\sqrt{u-1}}{\sqrt{u+1}} = k \Rightarrow \frac{xu\sqrt{u-1}}{\sqrt{u+1}} = C$$

$$\frac{x \cdot \frac{y}{x} \sqrt{\frac{y}{x} - 1}}{\sqrt{\frac{y}{x} + 1}} = C \Rightarrow \frac{y \sqrt{\frac{y}{x} - 1}}{\sqrt{\frac{y}{x} + 1}} = C \Rightarrow \frac{y \sqrt{y-x}}{\sqrt{y+x}} = C$$