Inspection shows that $(x^2-1)y''-2xy'+2y=0$ has $y_1=x$ as a first solution. Find another independent solution $y_2(x)$ by the method of reduction of order. [88 成大機械 2]

[解]令 $y=vx \Rightarrow y'=v'x+v,\ y''=v''x+2v'$

原式 $\Rightarrow (x^2-1)(v''x+2v')-2x(v'x+v)+2vx=0$

$x(x^2-1)v''-2v'=0$ 這是 v'的一階線性

$$v'=C_1e^{\int\frac{2}{x(x^2-1)}dx}=C_1e^{\int(\frac{1}{x+1}+\frac{1}{x-1}-\frac{2}{x})dx}=C_1e^{\ln(x+1)+\ln(x-1)-2\ln x}=C_1e^{\ln\frac{(x+1)(x-1)}{x^2}}=\frac{C_1(x^2-1)}{x^2}$$

$$v=\int\frac{C_1(x^2-1)}{x^2}dx+C_2=C_1\int(1-\frac{1}{x^2})dx+C_2=C_1(x+\frac{1}{x})+C_2$$

$$y=vx=C_1(x^2+1)+C_2x$$