

Solve the differential equation $y \frac{d^2 y}{dx^2} = \left(\frac{dy}{dx}\right)^2$. [87 台科大機械 1]

[解] 令 $p = y' \Rightarrow y'' = \frac{dy'}{dx} = \frac{dy'}{dy} \frac{dy}{dx} = p \frac{dp}{dy}$

原式 $\Rightarrow y \cdot p \frac{dp}{dy} = p^2 \Rightarrow \frac{dp}{p} = \frac{dy}{y}$ 此為 p 對 y 的一階微分方程式

$$\int \frac{dp}{p} = \int \frac{dy}{y} + k_1 \Rightarrow \ln p = \ln y + k_1 \Rightarrow \ln \frac{p}{y} = k_1 \Rightarrow \frac{p}{y} = C_1 \Rightarrow \frac{dy}{dx} = C_1 \Rightarrow \frac{dy}{y} = C_1 dx$$

$$\int \frac{dy}{y} = \int C_1 dx + k_2 \Rightarrow \ln y = C_1 x + k_2 \Rightarrow y = e^{C_1 x + k_2} = e^{C_1 x} \cdot e^{k_2} = C_2 e^{C_1 x}$$