Find the general solution of $x^2y'' - 2xy' + 2y = x^3\cos x$. [85 台科大機械 1]

[解]令 $x = e^t$

原式 $\Rightarrow (\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}) - 2\dfrac{dy}{dt} + 2y = e^{3t}\cos e^t \Rightarrow \dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = e^{3t}\cos e^t \cdots\cdots\cdots(\text{i})$

特徵方程式為 $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = 1, 2 \Rightarrow y_h = C_1e^t + C_2e^{2t}$

令 $y_p = u_1e^t + u_2e^{2t}$

$\dfrac{dy_p}{dt} = (u_1e^t + 2u_2e^{2t}) + (u_1'e^t + u_2'e^{2t})$

令 $u_1'e^t + u_2'e^{2t} = 0 \cdots\cdots\cdots(\text{ii})$

$\dfrac{d^2y_p}{dt^2} = (u_1e^t + 4u_2e^{2t}) + (u_1'e^t + 2u_2'e^{2t})$

$(\text{i}) \Rightarrow [(u_1e^t + 4u_2e^{2t}) + (u_1'e^t + 2u_2'e^{2t})] - 3(u_1e^t + 2u_2e^{2t}) + 2(u_1e^t + u_2e^{2t}) = e^{3t}\cos e^t$

$u_1'e^t + 2u_2'e^{2t} = e^{3t}\cos e^t \cdots\cdots\cdots(\text{iii})$

由(ii), (iii)知 $u_1' = \dfrac{\begin{vmatrix} 0 & e^{2t} \\ e^{3t}\cos e^t & 2e^{2t} \end{vmatrix}}{\begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}} = \dfrac{-e^{5t}\cos e^t}{e^{3t}} = -e^{2t}\cos e^t$

$u_2' = \dfrac{\begin{vmatrix} e^t & 0 \\ e^t & e^{3t}\cos e^t \end{vmatrix}}{\begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix}} = \dfrac{e^{4t}\cos e^t}{e^{3t}} = e^t\cos e^t$

$u_1 = -\int e^{2t}\cos e^t dt = -\int e^t d\sin e^t = -(e^t\sin e^t - \int \sin e^t de^t) = -e^t\sin e^t - \cos e^t$

$u_2 = \int e^t\cos e^t dt = \sin e^t$

解為 $y = y_h + y_p = C_1e^t + C_2e^{2t} - (e^t\sin e^t + \cos e^t)e^t + e^{2t}\sin e^t$

$\qquad = C_1e^t + C_2e^{2t} - e^t\cos e^t = C_1x + C_2x^2 - x\cos x$