Solve $yy'' + (y')^2 + y^2 = 0$. [85 交大機械丁 1]

[解]令 $y = e^z \Rightarrow y' = z'e^z \Rightarrow y'' = z''e^z + z'^2 e^z$

原式 $\Rightarrow e^z(z''e^z + z'^2 e^z) + (z'e^z)^2 + e^{2z} = 0 \Rightarrow e^{2z}(z'' + 2z'^2 + 1) = 0 \Rightarrow z'' + 2z'^2 + 1 = 0$

$$\frac{dz'}{2z'^2+1} + dx = 0 \Rightarrow \int \frac{dz'}{2z'^2+1} + \int dx = C_1 \Rightarrow \frac{1}{\sqrt{2}} \int \frac{d(\sqrt{2}z')}{(\sqrt{2}z')^2+1} + x = C_1$$

$$\frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}z') = C_1 - x \Rightarrow z' = \frac{1}{\sqrt{2}} \tan[\sqrt{2}(C_1 - x)] \Rightarrow \frac{dz}{dx} = \frac{1}{\sqrt{2}} \tan[\sqrt{2}(C_1 - x)]$$

$$\int dz = \int \frac{1}{\sqrt{2}} \tan[\sqrt{2}(C_1 - x)]dx + C_2 \Rightarrow z = -\frac{1}{2} \int \tan[\sqrt{2}(C_1 - x)]d[\sqrt{2}(C_1 - x)] + C_2$$

$$z = -\frac{1}{2} \ln \sec[\sqrt{2}(C_1 - x)] + C_2 \Rightarrow y = \ln\{C_2 - \frac{1}{2} \ln \sec[\sqrt{2}(C_1 - x)]\}$$