Find the general solution to the O.D.E. $(x^2-1)x^2\dfrac{d^2y}{dx^2}-(x^2+1)x\dfrac{dy}{dx}+(x^2+1)y=0$. [106 台大海洋二]

[解]$y=x$ 滿足原式 $\Rightarrow$ 令 $y=vx \Rightarrow y'=v'x+v,\ y''=v''x+2v'$

原式 $\Rightarrow (x^2-1)x^2(v''x+2v')-(x^2+1)x(v'x+v)+(x^2+1)vx=0$

$$x^3(x^2-1)v''+[2x^2(x^2-1)-x^2(x^2+1)]v'+[-x(x^2+1)+x(x^2+1)]v=0$$

$$x^3(x^2-1)v''+x^2(x^2-3)v'=0 \Rightarrow x(x^2-1)v''+(x^2-3)v'=0$$

$v''+\dfrac{x^2-3}{x(x^2-1)}v'=0$ 為 $v'$ 的一階線性

令 $\dfrac{x^2-3}{x(x^2-1)}=\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{x-1} \Rightarrow x^2-3=A(x+1)(x-1)+Bx(x-1)+Cx(x+1)$

$$x=0 \Rightarrow -3=-A \Rightarrow A=3$$

$$x=-1 \Rightarrow -2=2B \Rightarrow B=-1$$

$$x=1 \Rightarrow -2=2C \Rightarrow C=-1$$

$$v'=C_1e^{\int -\frac{x^2-3}{x(x^2-1)}\cdot dx}=C_1e^{\int(-\frac{3}{x}+\frac{1}{x+1}+\frac{1}{x-1})\cdot dx}=C_1e^{-3\ln x+\ln(x+1)+\ln(x-1)}=\frac{C_1(x^2-1)}{x^3}=C_1(\frac{1}{x}-\frac{1}{x^3})$$

$$v=\int C_1(\frac{1}{x}-\frac{1}{x^3})dx+C_2=C_1(\ln x+\frac{1}{2x^2})+C_2$$

$$y=vx=C_1(x\ln x+\frac{1}{2x})+C_2x$$