

試求  $x^2y'' - 3xy' + 4y = x^2$  的通解。[106 南大綠能 8]

[解] 令  $x = e^t \Rightarrow t = \ln x$  原式為

$$\frac{d}{dt} \left( \frac{d}{dt} - 1 \right) y - 3 \frac{dy}{dt} + 4y = e^{2t} \Rightarrow \frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 4y = e^{2t} \quad (i)$$

$$\text{特徵方程式 } \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2, 2 \Rightarrow y_h = (C_1 + C_2 t)e^{2t}$$

$$\text{令 } y_p = At^2 e^{2t} \Rightarrow y_p' = (2At^2 + 2At)e^{2t} \Rightarrow y_p'' = (4At^2 + 8At + 2A)e^{2t}$$

$$(i) \Rightarrow (4At^2 + 8At + 2A)e^{2t} - 4(2At^2 + 2At)e^{2t} + 4At^2 e^{2t} = e^{2t}$$

$$2Ae^{2t} = e^{2t} \Rightarrow A = \frac{1}{2}$$

$$y = y_h + y_p = (C_1 + C_2 t)e^{2t} + \frac{1}{2} t^2 e^{2t} = [C_1 + C_2 \ln x + \frac{1}{2} (\ln x)^2] x^2$$