Find the general solution of $x^3y^{(3)}-3x^2y''+6xy'-6y=\ln x$, $y(1)=4$, $y'(1)=y''(1)=1$. [106 中興電機乙丙丁 8(3)]

[解]令 $x=e^t \Rightarrow t=\ln x$

原式 $\Rightarrow \frac{d}{dt}(\frac{d}{dt}-1)(\frac{d}{dt}-2)y-3\frac{d}{dt}(\frac{d}{dt}-1)y+6\frac{dy}{dt}-6y=t$

$\frac{d^3y}{dt^3}-6\frac{d^2y}{dt^2}+11\frac{dy}{dt}-6y=t\cdots\cdots\cdots\text{(i)}$

特徵方程式為 $\lambda^3-6\lambda^2+11\lambda-6=0\Rightarrow(\lambda-1)(\lambda-2)(\lambda-3)=0\Rightarrow\lambda=1,2,3$

$y_h=C_1e^t+C_2e^{2t}+C_2e^{3t}$

令 $y_p=At+B\Rightarrow\frac{dy_p}{dt}=A,\ \frac{d^2y_p}{dt^2}=\frac{d^3y_p}{dt^3}=0$

$\text{(i)}\Rightarrow 0-6\cdot 0+11A-6(At+B)=t\Rightarrow A=-\frac{1}{6},\ B=-\frac{11}{36}$

解為 $y=y_h+y_p=C_1e^t+C_2e^{2t}+C_2e^{3t}-\frac{1}{6}t-\frac{11}{36}=C_1x+C_2x^2+C_2x^3-\frac{1}{6}\ln x-\frac{11}{36}$