Use the power series method to solve $xy''+y=0$. [105 中山材光 1]

[解]令 $y=x^r\sum_{m=0}^{\infty}A_m x^m \Rightarrow y'=\sum_{m=0}^{\infty}(m+r)A_m x^{m+r-1},\ y''=\sum_{m=0}^{\infty}(m+r)(m+r-1)A_m x^{m+r-2}$

原式 $\Rightarrow \sum_{m=0}^{\infty}(m+r)(m+r-1)A_m x^{m+r-1}+\sum_{m=0}^{\infty}A_m x^{m+r}=0$

$\sum_{m=0}^{\infty}(m+r)(m+r-1)A_m x^{m+r-1}+\sum_{m=1}^{\infty}A_{m-1}x^{m+r-1}=0$

$r(r-1)A_0x^{r-1}+\sum_{m=1}^{\infty}[(m+r)(m+r-1)A_m+A_{m-1}]x^{m+r-1}=0$

知 $r(r-1)=0,\ A_m=-\dfrac{A_{m-1}}{(m+r-1)(m+r)}\quad m=1,2,3,\cdots$

$r=1$時，$A_1=-\dfrac{A_0}{1\cdot 2},\ A_2=-\dfrac{A_1}{2\cdot 3}=\dfrac{A_0}{1\cdot 2^2\cdot 3},\ A_3=-\dfrac{A_0}{1\cdot 2^2\cdot 3^2\cdot 4},\cdots \Rightarrow A_m=(-1)^m\dfrac{A_0}{m!(m+1)!}$

令 $A_0=1\Rightarrow$ 得一解為 $y_1=\sum_{m=0}^{\infty}(-1)^m\dfrac{x^{m+1}}{m!(m+1)!}=x-\dfrac{x^2}{2}+\dfrac{x^3}{12}-\dfrac{x^4}{144}+\cdots$

$r=0$時，設第二個解 $y_2=C_2y_1\ln x+\sum_{m=0}^{\infty}B_m x^m$

$y_2'=C_2(y_1'\ln x+\dfrac{y_1}{x})+\sum_{m=0}^{\infty}mB_m x^{m-1},\ y_2''=C_2(y_1''\ln x+2\dfrac{y_1'}{x}-\dfrac{y_1}{x^2})+\sum_{m=0}^{\infty}m(m-1)B_m x^{m-2}$

代入原式 $\Rightarrow x[C_2(y_1''\ln x+2\dfrac{y_1'}{x}-\dfrac{y_1}{x^2})+\sum_{m=0}^{\infty}m(m-1)B_m x^{m-2}]+C_2y_1\ln x+\sum_{m=0}^{\infty}B_m x^m=0$

$C_2(2y_1'-\dfrac{y_1}{x})+\sum_{m=1}^{\infty}m(m-1)B_m x^{m-1}+\sum_{m=1}^{\infty}B_{m-1}x^{m-1}=0$

$C_2[2\sum_{m=0}^{\infty}(-1)^m\dfrac{m+1}{m!(m+1)!}x^m-\sum_{m=0}^{\infty}(-1)^m\dfrac{1}{m!(m+1)!}x^m]+\sum_{m=1}^{\infty}[m(m-1)B_m+B_{m-1}]x^{m-1}=0$

$C_2\sum_{m=0}^{\infty}(-1)^m\dfrac{2m+1}{m!(m+1)!}x^m+\sum_{m=1}^{\infty}[m(m-1)B_m+B_{m-1}]x^{m-1}=0$

$C_2\sum_{m=1}^{\infty}(-1)^{m-1}\dfrac{2m-1}{(m-1)!m!}x^{m-1}+\sum_{m=1}^{\infty}[m(m-1)B_m+B_{m-1}]x^{m-1}=0$

$C_2(-1)^{m-1}\dfrac{2m-1}{(m-1)!m!}+m(m-1)B_m+B_{m-1}=0,\ m=1,2,3,\cdots$

$C_2+B_0=0,\ -\dfrac{3}{2}C_2+2B_2+B_1=0,\ \dfrac{5}{12}C_2+6B_3+B_2=0,\cdots$

$B_0=-C_2,\ B_2=\dfrac{3}{4}C_2-\dfrac{1}{2}B_1,\ B_3=-\dfrac{5}{72}C_2-\dfrac{1}{6}B_2=-\dfrac{5}{72}C_2-\dfrac{1}{6}(\dfrac{3}{4}C_2-\dfrac{1}{2}B_1)=-\dfrac{7}{36}C_2+\dfrac{1}{12}B_1$

$y_2=C_2[(x-\dfrac{x^2}{2}+\dfrac{x^3}{12}-\dfrac{x^4}{144}+\cdots)\ln x+(-1+\dfrac{3}{4}x^2-\dfrac{7}{36}x^3+\cdots)]+B_1(x-\dfrac{x^2}{2}+\dfrac{x^3}{12}-\cdots)$

$y_2=C_2[y_1\ln x+(-1+\dfrac{3}{4}x^2-\dfrac{7}{36}x^3+\cdots)]+B_1y_1$

y_2即為此方程式的通解