試求解 $(2x+y-7)dx+(x+y-3)dy=0$ 之通解。[104 中山環工 1]

[解] $\begin{cases} 2x+y-7=0 \\ x+y-3=0 \end{cases} \Rightarrow x=4, y=-1$，令 $x=X+4, y=Y-1$，原式變為

$(2X+Y)dX+(X+Y)dY=0\cdots\cdots(i)$

令 $Y=uX \Rightarrow dY=Xdu+udX$

$(i) \Rightarrow (2X+uX)dX+(X+uX)(Xdu+udX)=0$

$(2+u)dX+(1+u)(Xdu+udX)=0$

$[(2+u)+(1+u)u]dX+(1+u)Xdu=0$

$(u^2+2u+2)dX+X(1+u)du=0 \Rightarrow \dfrac{dX}{X}+\dfrac{1+u}{u^2+2u+2}du=0$

$\displaystyle\int\dfrac{dX}{X}+\int\dfrac{1+u}{u^2+2u+2}du=k \Rightarrow \int\dfrac{dX}{X}+\dfrac{1}{2}\int\dfrac{d(u^2+2u+2)}{u^2+2u+2}=k$

$2\ln X+\ln(u^2+2u+2)=2k \Rightarrow \ln[X^2(u^2+2u+2)]=2k$

$X^2(u^2+2u+2)=C \Rightarrow (uX)^2+2X\cdot uX+2X^2=C$

$Y^2+2XY+2X^2=C \Rightarrow (y+1)^2+2(x-4)(y+1)+2(x-4)^2=C$