Consider the differential equation $y' = \dfrac{x+2y+6}{-2x+y-2}$. Find the general solution. [103 北科大電機 1]

[解]原式 $\Rightarrow (x+2y+6)dx + (2x-y+2)dy = 0$

$\begin{cases} x+2y+6=0 \\ 2x-y+2=0 \end{cases} \Rightarrow x=-2,\ y=-2$，令$X = x-2, Y = y-2$，原式變為

$(X+2Y)dX + (2X-Y)dY = 0 \cdots\cdots (\text{i})$

令$Y = uX \Rightarrow dY = Xdu + udX$

$(\text{i}) \Rightarrow (X+2uX)dX + (2X-uY)(Xdu+udX) = 0$

$(1+2u)dX + (2-u)(Xdu+udX) = 0$

$[(1+2u)+(2-u)u]dX + (2-u)Xdu = 0$

$(-u^2+4u+1)dX + X(2-u)du = 0 \Rightarrow \dfrac{dX}{X} + \dfrac{u-2}{u^2-4u-1}du = 0$

$\int \dfrac{dX}{X} + \int \dfrac{u-2}{u^2-4u-1}du = k \Rightarrow \int \dfrac{dX}{X} + \dfrac{1}{2}\int \dfrac{d(u^2-4u-1)}{u^2-4u-1} = k$

$2\ln X + \ln(u^2-4u-1) = 2k \Rightarrow \ln[X^2(u^2-4u-1)] = 2k$

$X^2(u^2-4u-1) = C \Rightarrow (uX)^2 - 4X \cdot uX - X^2 = C$

$Y^2 - 4XY - X^2 = C \Rightarrow (y-2)^2 - 4(x-2)(y-2) - (x-2)^2 = C$