$y_1(x)$ and $y_2(x)$ are the linearly-independent solutions of homogeneous-linear differential equation $y''+p(x)y'+q(x)y=0$. (a)Prove $y_p(x)=-y_1\int\frac{y_2r}{W}dx+y_2\int\frac{y_1r}{W}dx$ (where $W=y_1y'_2-y_2y'_1$) is the particular solution of nonhomogeneous-linear differential equation $y''+p(x)y'+q(x)y=r(x)$.

(b)Use (a) to solve $y''+y=\frac{1}{\cos x}$. [101 暨南電機(二)]

[解](a)令 $y_p=y_1u+y_2v\Rightarrow y'_p=(y'_1u+y'_2v)+(y_1u'+y_2v')$

令 $y_1u'+y_2v'=0\cdots\cdots\cdots$(i)

則 $y'_p=y'_1u+y'_2v\Rightarrow y''_p=(y''_1u+y''_2v)+(y'_1u'+y'_2v')$

代入原式 $\Rightarrow[(y''_1u+y''_2v)+(y'_1u'+y'_2v')]+p(y'_1u+y'_2v)+q(y_1u+y_2v)=r$

$(y''_1+py'_1+qy_1)u+(y''_2+py'_2+qy_2)v+(y'_1u'+y'_2v')=r\Rightarrow 0\cdot u+0\cdot v+(y'_1u'+y'_2v')=r$

$y'_1u'+y'_2v'=r\cdots\cdots\cdots$(ii)

由(i), (ii)得 $u'=\frac{\begin{vmatrix}0 & y_2\\ r & y'_2\end{vmatrix}}{\begin{vmatrix}y_1 & y_2\\ y'_1 & y'_2\end{vmatrix}}=\frac{-y_2r}{W}\Rightarrow u=-\int\frac{y_2r}{W}dx$

$v'=\frac{\begin{vmatrix}y_1 & 0\\ y' & r_2\end{vmatrix}}{\begin{vmatrix}y_1 & y_2\\ y'_1 & y'_2\end{vmatrix}}=\frac{y_1r}{W}\Rightarrow u=\int\frac{y_1r}{W}dx$

得 $y_p=-y_1\int\frac{y_2r}{W}dx+y_2\int\frac{y_1r}{W}dx$

(b)$y''+y=\frac{1}{\cos x}=\sec x$的齊性解為 $y_1=\cos x$及 $y_2=\sin x$

$W=y_1y'_2-y'_1y_2=\cos x\cdot\cos x-(-\sin x)\cdot\sin x=1$

$y_p=-\cos x\int\sin x\cdot\sec xdx+\sin x\int\cos x\cdot\sec xdx=-\cos x\int\tan xdx+\sin x\int 1\cdot dx$

$=-\cos x\cdot\ln|\sec x|+x\sin x$

解為 $y=C_1\cos x+C_2\sin x-\cos x\cdot\ln|\sec x|+x\sin x$